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ASSET BUBBLES AND OVERLAPPING GENERATIONS

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The first part of this paper considers the interaction between productive and nonproductive savings in a growing economy. It employs an overlapping generations model with capital accumulation and various types of rents, and gives necessary and sufficient conditions for the existence of an aggregate bubble. The second part is a series of thoughts on the definition, nature, and consequences of asset bubbles. First, it derives some implications of bubbles for tests of asset pricing. Second, it demonstrates the specificity of money as an asset and shows that there is a fundamental dichtotomy in its formalization. Third, it discusses inefficiencies of price bubbles. Fourth, it shows that the financial definition of a bubble is not satisfactory for some assets.

1. INTRODUCTION

THE VALUATION OF ASSETS is a long-standing problem in economics. Is there any rational foundation for actual prices of gold, stocks, land, or money itself? To answer this question finance theory generally assumes that the price of an asset equals the (expected) present discounted value of its dividends, i.e., its *market fundamental.* This view has, for example, been taken to test the causes of fluctuations of stock prices (see Leroy and Porter [**25**] and Shiller [**32, 33**]).

In a previous paper (Tirole [**35**]) I considered an economy with a finite number of infinitely lived traders and I showed that any asset must indeed be valued according to its market fundamental. This conclusion is robust to differential information and to the presence of short sales constraints.

This paper investigates whether the fundamentalist view of asset pricing extends to overlapping generations economies. As stated this question can readily be answered in the negative. Since Samuelson [**30**] developed his consumption loan model it has been well known that there exist economies in which money has a positive value in spite of the fact that it is intrinsically useless (i.e., its market fundamental is zero). In other words there can exist a *bubble* on money where a bubble is defined as the difference between the market price and the market fundamental.[2] However models in which money is the only store of value are peculiar. And it is sometimes conjectured that if traders hold a real asset there can not be price bubbles. The idea roughly runs as follows: If the long-run interest rate is nonpositive and if there exists a rent, i.e., an asset that distributes real dividends in each period, the market fundamental of this asset is infinite, so that this asset cannot be transferred between generations. On the other hand if the long-run interest rate is positive, the asset bubble—which must grow at the interest rate—eventually becomes so big that young generations cannot buy the asset. This loose reasoning has been made rigorous by Scheinkman [**31**]. As we will

[1] I am grateful to Olivier Blanchard, Peter Diamond, Stanley Fischer, David Kreps, Teodoro Millán, Jose Scheinkman, Hugo Sonnenschein, Philippe Weil, and especially to David Cass and Eric Maskin for helpful comments and discussions.

[2] Overlapping generations models with money have later been thoroughly developed by Gale [**17**], Cass-Okuno-Zilcha [**12**], Wallace [**38**], Hahn [**21**], Balasko-Shell [**4**], Grandmont [**19**], among others.

see, its validity however is limited to economies that do not grow. In a thought-provoking paper, Wallace [38] does consider a growing overlapping generations economy; he allows consumers to store a real good (but not to hold rents) and he shows the existence of monetary equilibria in which money serves no transaction purpose.

The goal of the first part of this paper is to complete and clarify a number of ideas within the framework of a model of capital accumulation based on Diamond's [15] celebrated analysis: Consumers hold stocks and other long run assets, including assets with a rent.[3] Section 3 describes the equilibrium set and studies the adjustment process. As bubbles crowd out productive savings and cannot grow faster than the economy, their existence is naturally shown to rely on the comparison between the asymptotic rates of growth and interest in the bubbleless economy. This comparison in turn depends on technology and preferences as well as the nature of rents. The latter's rate of growth and degree of ex-ante capitalization play a central role in the analysis. The main contribution of this section is the analysis of the implications of neo-classical production and especially various types of rents for the existence of bubbles. Section 4 suggests how institutions may operate a selection in the set of equilibria. It is shown that an arbitrarily small reserve requirement on an intrinsically useless asset leads to a given "asymptotically bubbly" path, i.e., a path in which the bubble does not become small relative to the economy.

The second part of the paper, "topics on bubbles," is a series of thoughts on the definition, nature, and consequences of asset bubbles. It is less technical than the first part; and most of it can be read independently.

Section 5 exhibits a general equilibrium with several assets in which the variance bounds obtained in Shiller [32] and Leroy-Porter [25] are reversed. In this equilibrium the whole path of consumption, savings, interest rates, and wages is deterministic. So are the market fundamentals of the assets. Asset prices, however, fluctuate over time. Furthermore they are negatively correlated at the aggregate level.

In Section 6 the valuation of money is examined in the light of the distinction between market fundamental and bubble. The market fundamental of money is equal to the present discounted value of transaction savings. It is shown that money differs from other assets in that (i) its market fundamental, and not only its price, depends on future prices, (ii) if money is expected to retain its transaction value in the future, there cannot exist a bubble on money. A corollary to the latter fact is that there is fundamental dichotomy in the formalization of money: Either it is essentially depicted as a store of value (bubble) as in Samuelson, or it is assumed to serve transaction purposes. The two representations are in the long run inconsistent.

Section 7 discusses potential inefficiencies of price bubbles at the micro economic level. First they may be costly to create; second they may prevent the

[3] Diamond was very careful not to introduce a long-run asset. However my treatment owes much to his analysis of the welfare effects of national debt. I also gained numerous insights for Section 3 from reading Wallace's [38] paper.

market fundamental of the corresponding assets from being exhausted. In both cases the market selection of "bubbly" assets may not be efficient.

Section 8 shows that the financial definition of market fundamental and bubble, which is adopted throughout the paper, is not always satisfactory. More subtle distinctions may be required for some assets.

Lastly Sections 9 and 10 conclude with some brief thoughts about what creates a bubble and about the use of the overlapping generations model to formalize asset pricing.

For conciseness the proofs of Propositions 1, 2, and 7 are given in the Appendix. The other (omitted) proofs can be found in Tirole [**35**].

PART I: EXISTENCE AND CHARACTERIZATION OF BUBBLES

2. THE BASIC MODEL

The model is based on Diamond's classic contribution. I refer the reader to the original paper for more details.

(a) *Consumers*: A consumer lives for two periods, but works only during the first. He supplies one unit of labor inelastically. Thus the labor force at time t, L_t, equals the number of young consumers at this date. We shall assume that the population grows at rate $n > 0$. Thus

(1) $$L_t = (1+n)^t L_0 = (1+n)^t, \quad \text{say.}$$

There is only one physical good per period.[4] The arguments of the consumer's unchanging utility function are his consumptions when young and old: $u(c^y, c^0)$. We will assume that both goods are normal and that the curvature of the utility function is bounded above and bounded away from zero. For the moment the unique income of a consumer born at t is his wage w_t. Let r_{t+1} denote the (real) interest rate at time $(t+1)$. Aggregate savings at time t can be written: $(1+n)^t s(w_t, r_{t+1})$ where s is the individual savings function.

(b) *Production*: There exists an unchanging twice continuously differentiable constant returns to scale technology:

(2) $$Y_t = F(K_t, L_t) = L_t f(k_t)$$

where Y_t is total output, K_t is the capital stock, and k_t is the capital stock per worker. Capital must be invested one period in advance. We will assume that $f(0) = 0$; $f'(0) = +\infty$; that marginal rates of substitution are decreasing; and that production using this technology is competitive. Then

(3) $$r_t = f'(k_t).$$

[4] The new features introduced by several physical goods are examined in Kehoe-Levine [**22, 23**]. In particular it should be stressed that the question of bubbles in general is distinct from that of indeterminacy of equilibria, although here they are identical.

As is well known our assumptions imply the existence of a downward sloping factor price frontier:

(4) $$w_t = \phi(r_t).$$

Lastly capital at date zero ($k_0 > 0$), and thus date-zero interest rate and wage, are given by history:

(5) $$r_0 = f'(k_0).$$

(c) *Equilibrium*: Given r_{t+1}, firms invest at time t so as to equalize the marginal productivity of capital and the interest rate (equation (3)). Let a_t be the difference between savings per capita and the level of capital stock per capita in the constant returns to scale sector. Thus we have

(6) $$r_{t+1} = f'\left(\frac{s(w_t, r_{t+1}) - a_t}{1+n}\right).$$

Following Diamond we assume that (6) defines a function

(7) $$r_{t+1} = \psi(w_t, a_t)$$

and that: $\psi_w < 0$ and $\psi_a > 0$.

We also make Diamond's stability assumption that the curves defined by $\{w = \Phi(r)\}$ and $\{r = \psi(w, 0)\}$ have a unique intersection and that the ψ curve is strictly steeper (in absolute value) than the Φ curve at this intersection in the (r, w) plane.[5]

Let $\bar{r}$ be defined by

(8) $$\bar{r} = \psi(\Phi(\bar{r}), 0).$$

In the case where $\forall t\, a_t = 0$, Diamond has shown that there exists a unique competitive equilibrium. In this equilibrium the interest rate converges to $\bar{r}$. The equilibrium path is efficient if $\bar{r} > n$ and inefficient if $\bar{r} < n$.[6] The interest rate $\bar{r}$ will play a crucial role in what follows. We now extend Diamond's model to include rents and bubbles.

First there exist assets that bring a real *rent* (dividend), such as a natural resource, land, paintings, and jewels (for their consumption value) or decreasing returns to scale technologies.[7] For simplicity we assume that the total rent in the economy is R units of real good per period. The market fundamental of the corresponding assets is for a sequence of real interest rates:

(9) $$F_t = R\left[\sum_{s=t+1}^{\infty} \frac{1}{(1+r_{t+1}) \cdots (1+r_s)}\right].$$

[5] Much of the analysis (but not all) can be carried out without this assumption; it then becomes lengthier.

[6] We do not consider the nongeneric case $\bar{r} = n$ for simplicity. This case requires the use of Cass's [11] efficiency criterion.

[7] As is well known the case of decreasing returns to scale production functions can be transformed into the constant returns case by defining appropriately "entrepreneurial ability" for each firm. This transformation amounts to introducing one input per decreasing returns to scale firm and interpreting the price of this input as the rent on the firm.

The market fundamental per capita, f_t, is defined by:

$$(10) \qquad f_t = \frac{F_t}{(1+n)^t}.$$

Note that f_t satisfies the following difference equation:

$$(11) \qquad f_{t+1} = \frac{1+r_{t+1}}{1+n} f_t - \frac{R}{(1+n)^{t+1}}.$$

Second consumers can invest in assets with a zero market fundamental. These assets are best thought of as pieces of paper, whatever their origin, and are called *bubbles*. In reality some of these "useless" pieces of paper are likely to pertain to the ownership of rents (in addition to their market fundamental) or to that of some firms producing with a constant returns to scale technology that is freely available. The aggregate bubble per capita is denoted b_t. Under perfect foresight the bubble must bear the same yield as capital, as, by definition, it does not distribute any dividend:

$$(12) \qquad b_{t+1} = \frac{1+r_{t+1}}{1+n} b_t.$$

Lastly bubbles must be positive:[8]

$$(13) \qquad b_t \geqslant 0.$$

The difference between savings per capita and capital stock per capita is then

$$(14) \qquad a_t = f_t + b_t.$$

It will be called *nonproductive savings*. This terminology may be a bit misleading as rents do bring dividends. It simply means that investment in such assets (rents and bubbles) does not increase capital accumulation, and thus future dividends in the economy.

DEFINITION 1. A *perfect foresight equilibrium* (henceforth equilibrium) is a sequence of interest rates (r_t), wages (w_t), bubbles per capita (b_t), market fundamentals of rents per capita (f_t), and nonproductive savings $(a_t \equiv f_t + b_t)$ satisfying (5), and for all t, (4), (7), (9), (10), (12), (13), and

$$(15) \qquad s(w_t, r_{t+1}) - f_t > b_t \geqslant 0.$$

DEFINITION 2. An equilibrium is *bubbly* if there exists t such that $b_t > 0$ (equivalently $\forall t\, b_t > 0$). It is *asymptotically bubbly* if the bubble *per capita* does not converge to zero.

[8] There cannot be a negative bubble on an asset that can be freely disposed of: If p_t and d_t denote its price and dividend at time t, the arbitrage equation requires that for any $T > 0$:

$$p_t = \sum_{\tau=1}^{T} \frac{d_{t+\tau}}{(1+r_{t+1}) \cdots (1+r_{t+\tau})} + \frac{p_{t+T+1}}{(1+r_{t+1}) \cdots (1+r_{t+T+1})}.$$

The first term on the right-hand side converges to the market fundamental (if it exists), and the second term is always nonnegative. Therefore the asset's price must exceed its market fundamental.

3. EXISTENCE OF BUBBLES

In the case where Diamond's (bubbleless and rentless) equilibrium is inefficient ($\bar{r}<n$), define $\hat{b}$ by:

$$n=f'\left(\frac{s(\Phi(n), n)-\hat{b}}{1+n}\right). \tag{16}$$

$\hat{b}$ is well-defined and is unique. Note that if the economy happens to start with initial capital level $f'^{-1}(n)$, there exists a rentless equilibrium with constant interest rate n and constant bubble per capita $\hat{b}$.

We can now state:

PROPOSITION 1:[9] (a) *If $\bar{r}>n$, there exists a unique equilibrium. This equilibrium is bubbleless and the interest rate converges to $\bar{r}$.*

(b) *If $0<\bar{r}<n$, there exists a maximum feasible bubble $\hat{b}_0>0$, such that:* (i) *for any b_0 in $[0, \hat{b}_0)$, there exists a unique equilibrium with initial bubble b_0. This equilibrium is asymptotically bubbleless and the interest rate converges to $\bar{r}$. The initial value of the rent f_0 decreases with the initial bubble b_0. (ii) there exists a unique equilibrium with initial bubble $\hat{b}_0$. The bubble per capita converges to $\hat{b}$ and the interest rate converges to n. $\hat{b}_0$ and the initial level of nonproductive savings $\hat{a}_0$ both increase with k_0.*

(c) *If $\bar{r}<0$, there exists no bubbleless equilibrium. There exists a unique bubbly equilibrium. It is asymptotically bubbly and the interest rate converges to n.*

Proposition 1, which is proved in the Appendix, states that the existence of bubbles is conditioned by the efficiency of the bubbleless equilibrium. Note that when the economy is asymptotically bubbleless, the total bubble $B_t\equiv(1+n)^t b_t$ grows indefinitely (as $\bar{r}>0$). However it becomes small relative to the economy.

The interpretation of Proposition 1 is simple indeed. Bubbles lower productive savings and thus increase the marginal productivity of capital and the interest rate. When the bubbleless and rentless economy is efficient, the interest rate in the same economy with bubbles and rents *a fortiori* must in the long run exceed the rate of population growth. But since the bubble B_t must grow at the rate of interest, it ends up growing faster than the resources of the economy. Thus bubbles are ruled out by wealth constraints, comforting the intuition given in the introduction. This intuition, however, turns out to be misleading when the economy in the long run can grow at a rate exceeding the interest rate, i.e., in the inefficient case.

Let us give a rough intuition for the proposition when there are *no rents* in the economy. The dynamic system can then be simply described by two difference

[9] In Proposition 3, we do not consider the nongeneric cases $\bar{r}=0$ and $\bar{r}=n$ for brevity.

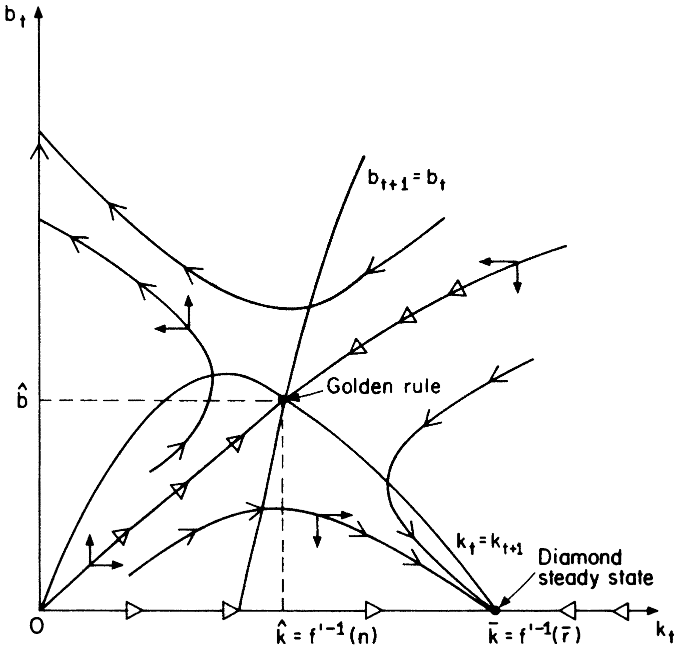

FIGURE 1.

equations in the per capita levels of capital and bubble:

$$b_{t+1} = \frac{1 + f'(k_{t+1})}{1+n} b_t, \tag{16'}$$

$$(1+n)k_{t+1} + b_t = s(f(k_t) - k_t f'(k_t), f'(k_{t+1})). \tag{16''}$$

To give an heuristic description of the behavior of this system, let us look at the corresponding phase diagram[10] (the continuous time representation offers only a convenient description of the properties found in Appendix 1; it is not meant to be a substitute for the discrete time analysis). The constant-capital-per-capita locus ("$k_{t+1} = k_t$") slopes negatively at the Diamond bubbleless steady state from our stability assumption. From (16′), (16″) and the assumption that savings increase with income, the constant-bubble-per-capita locus ("$b_{t+1} = b_t$") always slopes up. Figure 1, which is drawn for the inefficient case ($\bar{k} > \hat{k}$), depicts a few rational expectations paths (for a given path, only a countable number of points on this path will actually be observed because of the discrete time nature of our model).

The asymptotically bubbly path of Proposition 1 is the saddle path converging to the golden rule steady state in Figure 1. Along this path, the per capita levels

[10] The author is grateful to Philippe Weil for suggesting this diagrammatic interpretation.

of capital and bubble converge monotonically to their steady state values. Furthermore higher capital levels allow higher bubbles.[11] If the system starts under the saddle path, the equilibrium is asymptotically bubbleless, i.e., converges to the Diamond steady state. The system cannot start above the saddle path; if it did the bubble would inflate too fast, so that capital would become negative in finite time.

Let us reintroduce *rents*. We cannot employ the phase diagram any more (even with the usual caveat about its use to depict the evolution of a discrete time system). On the one hand the difference equation governing the evolution of nonproductive savings (the generalization of (16′)) is not autonomous any longer; see equation (11). On the other hand monotonic convergence to the steady state is not guaranteed. However, much of the previous intuition carries over. Indeed if the long-run rate of interest is strictly positive, the per capita value of rents becomes negligible and the dynamics then resemble those in Figure 1.

The long-run rate of interest in the bubbleless and rentless economy can be negative if capital depreciates. The interaction between rents and bubbles then becomes important, as shown by part (c) of the proposition. First there exists no bubbleless equilibrium. The intuition behind this fact roughly runs as follows: If the market fundamental of rents per capita f_t converges to zero, in the long run almost all the savings are used for capital accumulation. Thus the economy behaves asymptotically like the Diamond rentless and bubbleless economy and the interest rate converges to $\bar{r}<0$. But then the market fundamental of rents is infinite, a contradiction. Suppose now that f_t does not converge to zero. From (11) we know that this requires the interest rate to be relatively high in the long run (close to n). But then F_t is bounded and f_t converges to zero, a contradiction. Introducing bubbles, the same type of reasoning shows that there cannot be any asymptotically bubbleless equilibrium either. The only possible equilibrium is the asymptotically bubbly one. Thus if $\bar{r}<0$, bubbles are *necessary* for the existence of an equilibrium in an economy in which there exists an (arbitrarily small) rent. Under the condition that they do not become small relative to the economy they allow the interest rate to remain positive and bounded away from 0 in the long run and thus the market fundamental of rents to be well-defined.

Wallace [**38**] found that monetary equilibria (in which money is a pure bubble) exist if and only if the rate of population growth exceeds the coefficient of proportionality in the storage technology. Proposition 1 extends Wallace's analysis to the existence of rents (which, as shown by Scheinkman and the discussion above, may matter for the existence of bubbles). Furthermore its use of a production technology rather than a storage technology has some advantages. It allows one to analyze the capital adjustment path and the influence of the bubble on interest rates and market fundamental of rents.[12] And consumers always

[11] These two properties result from Lemmas 11 and 12 in the Appendix.

[12] The idea underlying the inverse relationship between the market fundamental f_0 and the bubble b_0 is the following: A high initial bubble leads to high interest rates. Thus the rents are discounted highly and their market fundamental is small. This argument, however, is not quite complete since a low market fundamental leads to high productive savings and thus to low interest rates. Proposition 1 shows that this latter effect is more than offset by the former.

hold several assets whereas in the equilibrium Wallace focuses on, which corresponds to the asymptotically bubbly equilibrium, consumers hold only one asset.[13]

We now briefly turn to the question of efficiency. An allocation is efficient if and only if it is not possible to improve the welfare of all generations (and this strictly for at least one of them). Proposition 2 states that, in the Diamond inefficient case, only the asymptotically bubbly equilibrium is efficient.

PROPOSITION 2: *If $\bar{r} < n$, then the asymptotically bubbleless equilibria are inefficient and the asymptotically bubbly equilibrium is efficient.*

The reader may rightly not feel satisfied with our formulation of rents. We have assumed that the total rent in the economy R remains constant over time, so that its value per capita becomes negligible if the long run rate of interest is strictly positive. This assumption, which was made to simplify the analysis, gives rise to the following objection: Imagine that rents per period grow at the rate of population growth, i.e., at the asymptotic rate of growth. If this is the case, a perfect foresight equilibrium must be efficient.[14] Otherwise the rent per period would grow at a rate exceeding the rate of interest and its market fundamental would be infinite. Hence the existence of such an asset would automatically put the economy in the efficient range and thus prevent bubbles.[15] As it is hard to rule out on a priori grounds that aggregate rents per period do not become small relative to the economy, I now argue that, because rents are created over time, bubbles are not necessarily inconsistent with rents per period growing as fast as the economy.

[13] Wallace justifies holding several assets by appealing to the need for diversification in a risky world. This justification is not necessary.

[14] This can be demonstrated using the definition of F_t and Theorem 5.6 in Balasko-Shell [**3**]. Informally, Cass's [**11**] criterion for production inefficiency is:

$$\sum_{t \geq 1} \frac{(1+r_1)\cdots(1+r_t)}{(1+n)^t} < +\infty;$$

this inequality implies that

$$F_t = R\left(\sum_{t \geq 1} \frac{(1+n)^t}{(1+r_1)\cdots(1+r_t)}\right) = +\infty,$$

which is impossible.

[15] Of course population growth is but one determinant of economic growth and certainly not the most powerful one. In Tirole [**35**] I give an example of an economy with Hicks-neutral *technological progress*; more precisely I specialize the basic model to Cobb-Douglas utility and production functions. I show that the no-rent economy behaves like an economy with no technological progress, but with fictitious rate of population growth $\tilde{n}$,

$$1+\tilde{n} = (1+n)(1+\mu)^{1/1-\gamma},$$

when n is the true rate of population growth, μ is the rate of technological progress, and γ is the share of capital in production.

In this case a rent that grows at the rate of population growth does not prevent bubbles. With technological progress the long-run rate of interest can exceed the rate of population growth without implying efficiency; what matters for bubbles is that it does not exceed the long-run rate of growth of the economy.

An important feature of rent creation is that most rents are *not capitalized* before their "creation." For example a painting to be created by a 21st century master cannot be sold in advance by the painter's forebears. Similarly patents cannot be granted for future inventions.[16] To formalize this idea without being overwhelmed by the intricacies of endogeneous rent creation, let us assume that each consumer is born with R units of real rents. The rest of the model is the same as before.

At time t, the level of existing real rents per capita, n_t, is:

$$n_t = \frac{R[1+(1+n)+\cdots+(1+n)^t]}{(1+n)^t} = \frac{(1+n)^{t+1}-1}{n(1+n)^t}R. \tag{17}$$

Note that $\lim n_{t\to\infty}\, n_t((1+n)R)/n$. Therefore rents do not become small relative to the economy contrary to the previous model.

Let f_t denote the value of one unit of real rent. Nonproductive savings equal $[b_t + n_t f_t]$ per capita. Generation t consumers have income $[w_t + Rf_t]$. Therefore (6) has to be replaced by (18):

$$r_{t+1} = f'\left(\frac{s(w_t + Rf_t, r_{t+1}) - (b_t + n_t f_t)}{1+n}\right). \tag{18}$$

We also have:

$$f_t = \frac{1}{1+r_{t+1}}(f_{t+1}+1). \tag{19}$$

DEFINITION 3: A perfect foresight equilibrium of the economy with (ex ante noncapitalized) rent creation is a sequence of interest rates (r_t), wages (w_t), bubbles per capita (b_t), market fundamental of rents (f_t), and levels of rents per capita (n_t) satisfying (4), (12), (13), (17), (18), (19), and

$$s(w_t + Rf_t, r_{t+1}) - n_t f_t > b_t \geqslant 0. \tag{20}$$

A complete analysis of this model is hard to derive. Mainly the consumers' portfolio composition fluctuates out of steady state. Also, income effects of rent creation complicate the study. Therefore we will content ourselves with an analysis of steady state behavior:

Let r^* be uniquely defined by

$$r^* = f'\left(\frac{s\left(\Phi(r^*) + \frac{R}{r^*}, r^*\right) - \frac{1+n}{n}\frac{R}{r^*}}{1+n}\right); \tag{21}$$

i.e., with a constant number of rents per capita $((1+n)R/n)$, the constant interest rate r^* is sustainable in the absence of bubbles (the value of one unit of rent per period being $1/r^*$).

[16] However for some firms, like IBM, the value of future patents (although their content is unknown) net of R&D costs might be capitalized in advance.

PROPOSITION 3: *There exists an (asymptotically) bubbly steady state of the economy with rent creation if and only if $r^* < n$. The steady bubble per capita, $\hat{b}$, is then given by*

$$n = f'\left(\frac{s\left(\Phi(n) + \frac{R}{n}, n\right) - \frac{1+n}{n}\frac{R}{n} - \hat{b}}{1+n}\right). \tag{22}$$

It is also easy to show that, under our assumptions, $\hat{b}$ decreases with R: Intensive rent creation crowds out bubbles. Indeed there exists a maximum rate of rent creation consistent with bubbles.[17]

The moral of our model of rent creation is that *one can build economies in which rents per period do not become small relative to the economy and there is still scope for bubbles.* In our model the absence of ex-ante capitalization is crucial to this conclusion: At any moment of time most rents still remain to be created and thus do not necessarily crowd out current bubbles fully.

4. INSTITUTIONAL BACKING OF AN ASSET

The idea of backing an asset has recently been formalized in the monetary literature. It is now well-known that when money is the only asset and has no transaction value there in general exists a continuum of inefficient equilibria, in which the economy is asymptotically nonmonetary.

Wallace [**38**] and Millán [**28**] have looked at whether government intervention may prevent this kind of equilibrium. In Wallace the government stands ready to buy any amount of money if the price of money in terms of real good falls below some floor level. Thus under perfect foresight consumers know that the price of money won't fall to zero and thus select an asymptotically monetary equilibrium. Millán requires that consumers hold at each instant (at least) a fixed fraction of their total assets in money. They are indemnified for any loss on their money reserves relative to the market interest rate. To operate these transfers the government levies a lump-sum tax on the young generation. Again the value of money cannot converge to zero since otherwise the consumers would not be able to meet the reserve requirement. In both papers government intervention ensures efficiency.

Wallace's scheme cannot readily be applied to our economy. Even if the economy eventually becomes bubbleless, the real value of the aggregate bubble grows over time if the interest rate is positive. Thus a more sophisticated scheme than a single floor price would be required to ensure efficiency. Millán's scheme however can readily be applied to our framework. Our approach differs from Millán's in that consumers are not compensated for a loss in interest due to the reserve requirement.

[17] Whereas, in the model where rents are capitalized at date 0, arbitrarily high rents do not crowd out bubbles totally (see Proposition 1).

For simplicity assume that consumers can hold real capital and intrinsically useless assets. Rents could easily be introduced into this model. One of the useless assets, that we call gold rather than money, for reasons that will become clear in Section 6,[18,19] is "backed." Consumers must invest at least a fraction ξ of their savings in gold. The gold stock is constant. The total bubble per capita, b_t, can be decomposed into two bubbles: g_t, the bubble per capita on gold, and $d_t = b_t - g_t$, the bubble per capita on other useless assets. d_t must satisfy the arbitrage equation

$$d_{t+1} = \frac{1+r_{t+1}}{1+n} d_t \tag{23}$$

whereas g_t satisfies

$$g_{t+1} \leqslant \frac{1+r_{t+1}}{1+n} g_t. \tag{24}$$

If (24) is satisfied with a strict inequality we will say that there is a *crash* on gold at time t. Note that whether there is a crash or not the interest rate facing the consumer is:

$$(1-\xi)r_{t+1} + \xi\left[\frac{g_{t+1}}{g_t}(1+n) - 1\right] \equiv \tilde{r}_{t+1}.$$

Thus (6) becomes:

$$r_{t+1} = f'\left(\frac{s(w_t, \tilde{r}_{t+1}) - d_t - g_t}{1+n}\right). \tag{25}$$

Lastly we require that reserve requirements be met:

$$g_t \geqslant \xi s(w_t, \tilde{r}_{t+1}). \tag{26}$$

Notice that there is a complementarity slackness relationship between equations (24) and (26): For the consumer to hold more gold than necessary (inequality in (26)), gold must bear the same interest rate as capital (equality in (24)). And conversely if $\tilde{r}_{t+1} < r_{t+1}$ (inequality in (24)), then $g_t = \xi s(w_t, \tilde{r}_{t+1})$.

DEFINITION 4: A perfect foresight equilibrium of the economy with reserve requirements is a sequence of interest rates (r_t), wages (w_t), bubbles per capita on backed and unbacked assets (g_t and d_t) satisfying (5), (4), (23), (24), (25), (26), the complementarity slackness condition, and

$$s(w_t, \tilde{r}_{t+1}) > d_t + g_t > 0, \; d_t \geqslant 0, \; g_t > 0.$$

Note that the dynamics of the economy are given by a system of multivalued difference equations. Therefore uniqueness of an equilibrium path is uncertain.

[18] This section however is also relevant when money is used for transactions. A reserve requirement on money similarly operates a selection in the set of equilibria.

[19] As our only goal is to illustrate the working of a class of models, no attempt is made at giving a realistic description of a banking system.

As a matter of fact we will be concerned with approximate uniqueness: Is there a reference path such that any equilibrium path is "close to" this path?

To simplify the study we shall assume throughout this section that the consumer's utility function is Cobb-Douglas:

$$u(c^y, c^0) = \beta \log c^y + (1-\beta) \log c^0. \tag{27}$$

This assumption implies that savings depend only on the current wage, which simplifies the derivations considerably. I conjecture that the properties derived below hold for more general utility functions.

First we consider existence and long-run behavior of an equilibrium path. Let $\hat{b}$ be defined, as in Section 3 (equation (16)), as the bubble per capita such that an economy with initial bubble $\hat{b}$ and initial capital stock $f'^{-1}(n)$ stays in steady state. Let $\hat{\xi}$ be defined by:

$$\hat{\xi} = \frac{\hat{b}}{s(\Phi(n), n)} = \frac{\hat{b}}{(1-\beta)\hat{w}}. \tag{28}$$

$\hat{\xi}$ is the fraction of savings in the bubbly steady state that is held in useless assets (notice that if $\bar{r} \geq n$, then $\hat{b} = 0$ and thus $\hat{\xi} = 0$).

Lastly for $\xi > \hat{\xi}$, define $r^*(\xi) > n$ by

$$r^*(\xi) = f'\left(\frac{\Phi(r^*(\xi))(1-\beta)(1-\xi)}{1+n}\right).$$

Notice that $dr^*/d\xi > 0$.

We can now state Proposition 4 (in Proposition 4, we do not as usual consider the borderline case $\xi = \hat{\xi}$ for brevity. The proof of this proposition, as well as those of Propositions 5 and 6 can be found in Tirole [**35**].

PROPOSITION 4 (Asymptotic Uniqueness): *Assume that the utility function is Cobb-Douglas. For any ξ less than one there exists an equilibrium of the economy with reserve requirement.*

(a) *If $\xi < \hat{\xi}$, the equilibrium interest rate converges to n (golden rule). In the long run there is no crash on gold. Bubbles on assets other than gold can exist. The asymptotic aggregate bubble per capita is $\hat{b}$.*

(b) *If $\xi > \hat{\xi}$, the equilibrium interest rate converges to $r^*(\xi) > n$. In the long run there is a crash on gold in each period. Bubbles on unbacked assets do not exist. The asymptotic bubble per capita on gold is $\xi\Phi(r^*(\xi))(1-\beta)$.*

Proposition 4 shows that even if the equilibrium need not be unique, its long-run properties are uniquely defined. They depend crucially on whether the bubble in the bubbly steady state of the unbacked economy meets the reserve requirement. If it does, the economy behaves in the long run as in the asymptotically bubbly equilibrium of the unbacked economy. If it doesn't, then the long-run interest rate exceeds its golden rule level and a perpetual crash prevents the bubble on gold from overinflating. And the high interest rates rule out bubbles on unbacked assets.

Now we consider small reserve requirements and we show that the whole equilibrium path, and not only its long-run behavior, is unique in an approximate sense. Let b_t be the bubble per capita on an equilibrium path. Let $\hat{b}_t$ denote the bubble per capita in the asymptotically bubbly equilibrium of the unbacked economy (see Section 3; $\hat{b}_t$ converges to $\hat{b}$).

PROPOSITION 5 (Approximate Uniqueness): *Assume that the utility function is Cobb-Douglas. There exists a positive function $\alpha(\xi)$ such that*:

(i) $$\forall t \quad (1-\alpha(\xi))\hat{b}_t \leqslant b_t \leqslant \left(\frac{1}{1-\alpha(\xi)}\right)\hat{b}_t,$$

(ii) $$\lim_{\xi \to 0} \alpha(\xi) = 0.$$

Similar uniform bounds can be derived for wages, capital levels, and interest rates.

A corollary of Proposition 5 is that if the backed asset is the only bubble in the economy, the equilibrium path is unique and is the asymptotically bubbly path (the reserve requirement can never be binding as $b_t \geqslant (1-\alpha(\xi))\hat{b}_t > (\xi/1-\alpha(\xi))\hat{s}_t \geqslant \xi s_t$).

Proposition 5 implies that an arbitrarily small reserve requirement leads to the asymptotically bubbly path of the unbacked economy. In particular the asymptotically bubbleless equilibria described in Sections 3 and 4 are not robust to small reserve requirements whereas the asymptotically bubbly one is.

Lastly we study the efficiency properties of the equilibria with reserve requirements. As Proposition 5 shows, the existence of a reserve requirement in a sense solves the long run efficiency problem by ruling out asymptotically bubbleless paths in the inefficient case. However it creates a short run inefficiency by introducing a wedge between marginal rates of transformation and substitution when the constraint is binding.[20] Intuitively this short-run inefficiency should not matter much for small reserve requirements as $\tilde{r}_t$ is close to r_t. This intuition is formalized in Proposition 6.

PROPOSITION 6 (Approximate Efficiency): *Let $\tilde{u}_t$ denote generation t's utility in an equilibrium with reserve requirement, and let $\hat{u}_t$ denote generation t's utility in the (efficient) asymptotically bubbly path for the economy without reserve requirement. The sequence $\{\tilde{u}_t\}$ converges uniformly over time toward the sequence $\{\hat{u}_t\}$ when the reserve requirement converges to zero.*

Proposition 6 is a straightforward consequence of the uniform convergence of the wage to $\hat{w}_t$ (Proposition 5) and of the interest rate $\tilde{r}_t$ faced by consumers to $\hat{r}_t$ (Proposition 5 and definition of $\tilde{r}_t$).

[20] This differs from Millán's Pareto efficiency result. Millán posits an interest payments' subsidy financed through lump-sum taxation.

PART II: TOPICS ON BUBBLES

5. TESTS OF FLUCTUATIONS OF STOCK PRICES

The valuation of stocks has recently been the topic of several researches (see, e.g., Grossman-Shiller [**20**], Leroy-Porter [**25**], and Shiller [**32, 33**]). These studies show that the variance bounds based on the assumption that stock prices equal their market fundamental are systematically violated. The simplest such bound is obtained by noticing that the variance of the market price should not exceed that of the market fundamental. Simply by looking at Figures 1 and 2 in Shiller [**32**], this inequality is not satisfied. Grossman and Shiller try to explain this phenomenon by time-varying real interest rates.

In this section I simply want to point out that this kind of variance bound does not hold if bubbles exist, and that there may exist a substitution effect between the different asset prices.

Consider the economy described in Section 3, and the corresponding asymptotically bubbly path (the latter exists if the bubbleless economy is inefficient, and is approximately unique if there exists an arbitrarily small reserve requirement on an intrinsically useless asset (see Section 4)). The paths of consumption, real capital, wages, interest rates, and aggregate bubble per capita are given. However, as we noticed before, *the decomposition of the aggregate bubble is not given.* For simplicity assume that consumers can hold two assets: stocks and gold. Gold is intrinsically useless. There is a bubble on both assets: B_t^s and B_t^g. The aggregate bubble $B_t = B_t^s + B_t^g$ is deterministic. But the bubble on gold follows a discounted martingale:

$$E(B_{t+1}^g) = (1 + r_{t+1})B_t^g.$$ [21]

So does the bubble on stocks. Note that this stochastic process can be created by elements irrelevant to the economy, i.e., sunspots.[22]

Consumers are perfectly insured by holding the representative portfolio. Therefore the asymptotically bubbly equilibrium described in Section 3 is still an equilibrium of the economy with "*bubble substitution*," i.e., an economy in which there is a random transfer of a bubble from one asset to the other.

Now this equilibrium has some interesting properties: Interest rates are deterministic and so is the market fundamental of stocks. However the bubble on stocks and therefore the price of stocks can fluctuate wildly.[23] Thus the variance bound mentioned above is reversed. Also the asset prices are negatively correlated.

This remark of course matters for the tests of fluctuations of stock prices. That the Schiller-Leroy-Porter variance bounds can be violated in the presence of

[21] Note that this martingale must satisfy some constraint: The bubble on gold must not exceed the aggregate bubble (and possibly must satisfy the reserve requirement if any).

[22] Sunspot models have been developed by Azariadis [**1**], Cass-Shell [**13**], and Azariadis-Guesnerie [**2**]. I think that sunspot phenomena can be particularly relevant to the study of asset bubbles (although they have a much greater generality; for example they do not rely on an infinite horizon).

[23] Note that by the martingale convergence theorem (see, e.g., Breiman [7, p. 89]) the bubble on gold must eventually converge.

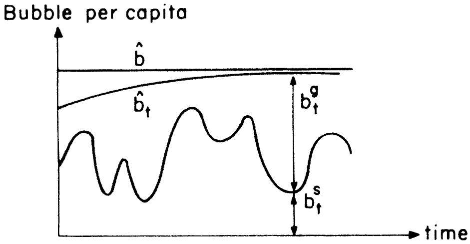

FIGURE 2—Bubble substitution.

bubbles is also noticed in a partial equilibrium set-up by Blanchard-Watson [**6**], who examine the econometric consequences of this fact.

The presence of bubbles may well account for the price fluctuations of some volatile assets like gold. Similarly the high variance of stock prices seems to suggest the existence of a bubble on stocks. This however may be inconsistent with the belief held by part of the market that the "right" price for a stock is given by the firm-foundation theory (according to Malkiel [**27**, p. 97], "perhaps 90 per cent of the Wall Street security analysts consider themselves fundamentalists"[24]). This important topic does warrant further analysis.

Our other conclusion, the negative correlation over the *whole* set of asset prices, is worth studying both from a theoretical and from an empirical point of view. It implies that, in an economy without aggregate uncertainty, any set of assets is negatively correlated with the complementary set in the consumers' portfolio. The empirical tests will be hard to conduct, because, for a number of assets, one is missing data on quantities (gold, jewels, etc.), or prices (paintings, . . .) or both.

Note that the hypothesis of overall negative correlation is at variance with the Grossman-Shiller explanation of fluctuations in stock prices: If the latter are to be explained by fluctuating rates of interest, all asset prices ought to be positively correlated (which is not the case). Theories of stochastic *aggregate* bubble (see Azariadis [**1**], Azariadis-Guesnerie [**2**], and Weil [**39**]) also lead to the opposite conclusion. Reality may indeed reflect the two forces, and only empirical investigations will be able to determine the degree of correlation of asset prices.

[24] A second force that may pull the stock prices closer to these market fundamentals is the scarcity requirement (see Section 9). Assume that, because of lack of "focal stocks," the bubble must affect the whole stock market. If there is such a bubble, there is a tremendous incentive to create new firms, i.e., sell a bubble as well as a market fundamental. This would violate the scarcity condition.

What about existing firms? Can a firm the stock price of which contains a bubble make money by issuing new shares? This question is hard to answer. As we have shown earlier individual bubbles are certainly not uniquely defined by economic parameters. It is then likely that perfect foresight equilibria—or rational expectations equilibria in a more sophisticated model—must be associated with (focal) rules that consumers follow. If one of these rules says that the stock price of a firm must decrease in proportion of the number of shares issued, then in our set-up there is no scope for profit for the firm by issuing new shares (note that this rule holds in a bubbleless economy). Such a rule may support a bubble by decreasing the incentive to expend the asset.

6. THE MARKET FUNDAMENTAL OF MONEY AND TESTS OF MONETARY BUBBLES

In this section I consider monetary models in the light of the distinction between market fundamental and price bubble. Assume that money is held for transaction purposes. To simplify, the utility of a consumer born at time t depends on consumption when young and old and on real money held when young: $u(c^y, c^0, p_t M^y)$, where p_t is the price of money in terms of good and M^y is the amount of money held when young. This clearly is not a satisfactory model of transaction demand, but it will suffice to illustrate our purpose. The consumer maximizes his utility subject to the constraints $c^y = w_t - p_t M^y - h$ and $c^0 = (1+r_{t+1})h + p_{t+1}M^y$, where w_t and h denote his first period wage and nonmonetary savings (the latter being a choice variable). From the first order conditions, the price of money in terms of good can be decomposed into two terms:

$$(29) \qquad p_t = \sum_{\tau=0}^{\infty} \left[\frac{p_{t+\tau} u_m^{t+\tau}}{u_y^{t+\tau}}\right]\left[\frac{1}{(1+r_{t+1}) \cdots (1+r_{t+\tau})}\right] + e_t$$

where

$$(30) \qquad e_{t+1} = (1+r_{t+1})\, e_t.$$

Thus e_t is a bubble. The market fundamental of one unit of money is the first term on the right-hand side of (29). $u_m^{t+\tau}/u_y^{t+\tau}$ is the marginal rate of substitution between money and consumption when young. Thus $\{p_{t+\tau} u_m^{t+\tau}/u_y^{t+\tau}\}$ is the marginal utility of one unit of money at time $(t+\tau)$. We thus check that the market fundamental is equal to the present discounted value of the "dividends" distributed by the asset.

Equation (29) is instructive in another respect. Contrary to the case of the assets considered in Section 4, the monetary market fundamental is not defined solely by the sequence of real interest rates. Its dividend depends on its price. This suggests a possible multiplicity of market fundamentals. And indeed several authors have shown that, in an economy in which money is the only asset and is used for transaction purposes, there can be a continuum of *bubbleless* monetary equilibria (Brock-Scheinkman [**9**], Obstfeld-Rogoff [**29**], Wilson [**40**, **41**]; and Geanakoplos-Polemarchakis [**18**] in an economy with production). In view of these contributions there is no point proving multiplicity of market fundamentals of money here. But I want to insist on the fact that this multiplicity has nothing to do with overlapping generations. Indeed it can be obtained in a model in which agents live forever (see Wilson [**40**, **41**], Obstfeld-Rogoff [**27**]). Under some conditions on how essential money is, one can rule out multiplicity of its market fundamental (see, e.g., Obstfeld-Rogoff). A reserve requirement on money or the need to pay taxes in money would, for example, serve the purpose. We now examine the possibility of bubbles on money.

Flood and Garber [**16**] give the first tests of asset bubbles in a monetary context (the 1922-23 German hyperinflation). To this purpose they use a model of *money* *demand à la Cagan.* As is usually done for other assets, they iterate the Cagan first-order difference equation to decompose solutions in a forward solution, that they call market fundamental, and nonstationary solutions, that they call bubbles. Their approach raises several theoretical problems. The ad-hoc demand function for money is not an innocent assumption. It implies that the "market fundamental" of money can be obtained by simply knowing the rates of growth of money (as well as the expected shocks to the money demand equation). Thus their market fundamental is unique without any institutional restriction. What (29) shows on the contrary is that the market fundamental of money in general depends on the whole path of prices (to this extent money is a very special asset). A more basic criticism to their approach is that, to be able to identify the market fundamental, one must collect information about expected prices and then compute the real value of money in each period. The latter can only be obtained by using a satisfactory model of transaction demand, which we do not possess. A corollary is that monetary bubbles are hard to identify. And indeed the Flood–Garber nonstationary solutions are not bubbles in the financial acceptance of the term. They do not grow at the rate of interest.

In view of the difficulties involved in empirically estimating money bubbles, it may be worth pursuing the theoretical foundations of such bubbles. The following proposition is such an attempt. Let us first make the following assumption:

ASSUMPTION A: (*i*) *First order partial derivatives of the utility function* $u(c^y, c^0, m)$ *are strictly positive.* (*ii*) *If* c^y/m *does not converge to zero, then* u_m/u_y *does not converge to zero.* (*iii*) *There exists* $\varepsilon > 0$ *such that the propensity to save is bounded above by* $(1-\varepsilon)$.[25]

PROPOSITION 7: *Under Assumption A, there cannot exist a bubble on money.*

Proposition 7 (proved in the Appendix) also holds when money is backed through a reserve requirement. It again demonstrates that money is a very special asset. The reason is that its market fundamental depends on its future prices. This puts constraints on money price dynamics. Before studying the implications of this proposition, we discuss its robustness.

A more general formula for the market fundamental of one unit of *nominal* money is the following:

$$(31) \qquad f_t = \sum_{\tau=0}^{\infty} \frac{p_{t+\tau} T_{t+\tau}}{(1+r_{t+1}) \cdots (1+r_{t+\tau})}$$

where $p_{t+\tau}$ denotes the price of money in terms of good and $T_{t+\tau}$ denotes the transaction savings or shadow value associated with one unit of *real* balances (at time $(t+\tau)$). Imagine there is a strictly positive bubble e_t per unit of nominal money. Then $p_{t+\tau} \geqslant e_{t+\tau} = (1+r_{t+1}) \cdots (1+r_{t+\tau}) e_t$. This implies $f_t \geqslant (\sum_{\tau=0}^{\infty} T_{t+\tau})\, e_t$.

[25] Assumption A is, for example, satisfied by Cobb–Douglas utility functions.

We thus conclude that *if $T_{t+\tau}$ does not converge to zero*[26] *as τ goes to infinity, there cannot exist a bubble on money.* Note that this result is very general as there is no assumption on the rate of growth of money supply or on the underlying economy.

Two other applications can be made of this result: (a) Clower constraints: Assume that money does not enter the utility function, but is necessary to buy goods when young: $c^y \leq p_t M^y$ (this kind of constraint is hard to justify in two-period life economies; but the result could easily be extended to economies in which traders live during more than two periods). Let λ_t denote the multiplier associated with this constraint. The reader can check that if the transaction constraint remains binding in the long run, i.e., if the shadow value of real money expressed in terms of good $\{\lambda_t/u_y^t\}$ does not converge to zero, there cannot exist a bubble on money.

(b) Similarly if money is needed to pay an income tax (as in Starr [**34**]), and if this financing constraint remains binding in the long run, there cannot exist a bubble on money.

The following conclusion emerges from the analysis: There are two ways of giving value to money, and these are inconsistent, while they are not for other assets:

(i) The "*fundamentalist view*": Money is held to finance transactions (or to pay taxes or to satisfy a reserve requirement). To this purpose, money must, of course, be a store of value. However, it is not held for speculative purposes as there is no bubble on money.

(ii) The "*bubbly view*": Money is a pure store of value à la Samuelson. It does not serve any transaction purpose at least in the long run.[27] This view implies (in the long run) that the price of money (bubble) grows at the real rate of interest, and that money is held entirely for speculation.

Preliminary analysis of the fundamentalist and the bubbly views show that they have a number of features in common. First both allow a multiplicity of equilibria. Second a number of equilibria can be ruled out by institutional constraints (reserve requirements, money-financed taxes, etc.) or (less primitive) assumptions about how money is essential for transactions. Because of these similarities and because the traditional notion of market fundamental is here somewhat blurred by the influence of the market price on the dividend, one may be tempted to treat the distinction between these two inconsistent views as being purely semantic.

However the two views differ in many of their implications. First, only the fundamentalist view can explain the rate of return dominance of money by other assets. Second only this view seems to be consistent with money keeping some value under intensive creation (see footnote 31). Third the fundamentalist view can also account for inflations while the bubbly view cannot if the real rate of return is nonnegative. Fourth the two views imply different degrees of substitution

[26] Actually it suffices that it does not converge "too fast."

[27] Temporarily money may yield a rate of interest lower than the market rate because it is used for transactions. But this effect must disappear sufficiently fast to allow a bubble.

of money with other assets in the consumers' portfolios (see Section 5 for a start on this idea). Fifth, for modelling purposes, the fundamentalist view does not require overlapping generations while the bubbly view does.

7. POTENTIAL INEFFICIENCIES OF PRICE BUBBLES

Until now we have drawn a favorable picture of price bubbles. They help transfer goods from the young generation to the old and to that extent are the private counterpart to national debt à la Diamond.[28] Indeed when the bubbleless equilibrium is inefficient, the asymptotically bubbly equilibrium allows the economy to reach efficiency. Here we qualify this somewhat rosy picture by considering two kinds of inefficiency potentially associated with price bubbles. One type of inefficiency has to do with the cost of creating a bubbly asset. The other is the problem of the nonexhaustion of the market fundamental of a bubbly asset.

(a) *Costly bubble creation.* Imagine that there is a bubble on an asset which is intrinsically useless, say. Then it may pay to create new units of the asset, even if creation is costly (one can for instance envision the asset as being mined ore). This creation however does not create any real wealth in the economy, which suggests that the equilibrium is inefficient. In Tirole [**35**] it is shown that such an equilibrium path not only is inefficient, but also is Pareto dominated by some other perfect foresight equilibrium path in which creation of this asset is prohibited. The idea is that the agents who created the asset might as well save the creation costs, and issue pieces of paper with the same speculative value.

(b) *Nonexhaustion of a market fundamental.* Consider the basic model described in Section 2 and assume that asset bubbles can exist. The produced good is called a schmoo. It is well-known that produced schmoos are white. However, by a fluke, firms at date zero produce a few black schmoos. Everyone agrees that this productive miracle will never occur again. Also it turns out that black and white schmoos are perfect substitutes in the consumers' utility function, and that moreover black schmoos are costlessly storable. At date zero everyone thinks that due to their scarcity the black schmoos should be priced above the white schmoos. The excess of their price to one is low enough so that a perfect foresight equilibrium prevails. In such an equilibrium the black schmoos are never consumed if interest rates are positive because their price in terms of white schmoos always exceeds one.

It is clear that such an equilibrium is infficient. Imagine that the old generation at date zero issues pieces of paper with the same value as that of black schmoos, and consumes the black schmoos. The rest of the path is unchanged and the old generation is better off. This phenomenon is nothing but the well-known problem of potential inefficiency of private extraction of a nonrenewable resource (see, e.g., Dasgupta–Heal [**14**, Ch. 8]). In this alternative formulation oil (our black

[28] National debt does not contain a bubble since it is valued at its market fundamental. The reason why it acts as a bubble is that it is rolled over and never repaid.

schmoos) is not exhausted in the long run because the owners of oil fields are concerned with future capital gains (bubbles on black schmoos). Section 3 gives the conditions under which this kind of phenomenon can arise.

8. IS THE DISTINCTION MARKET FUNDAMENTAL/BUBBLE SATISFACTORY?

At the beginning of the paper, we adopted the financial definition of the bubble on an asset as being the difference between its price and the present discounted value of its dividends. And we said that, from the arbitrage condition, the bubble grows at the rate of interest. It turns out that, in some cases, the usual notion of market fundamental and bubble is not fully satisfactory (although these magnitudes can still be defined). There are two reasons why it may be so. The first is related to an illusion in bubble accounting, and can easily be remedied. The second is potentially more damaging for the dichotomy and shows that a more subtle analysis is required in some cases.

(a) *A pitfall in bubble accounting*: Assume that there exists a partially unmined natural resource in limited supply. Extracted ore is completely useless, i.e., its market fundamental is zero. Consumers invest in extracted ore and in stocks of the mines (among other assets). Thus there is a bubble on extracted ore. The market fundamental associated with the ownership of mines is the present discounted value of dividends, i.e., of ore that will be extracted and sold on the extracted ore market. This market fundamental is in general strictly positive.

A simple and instructive result is that, if there are no extraction costs, the sum of the current value of extracted ore and the value of stocks of the mines grows at the real rate of interest, i.e., is a bubble.[29] What happens is that the so-called market fundamental associated with the ownership of mines is a bubble itself as it *feeds off another bubble*, that on extracted ore. The moral is that, to obtain the aggregate bubble, one must add "direct" and "indirect" bubbles.

(b) *Backing of an asset and the financial dichotomy*: (i) Let us first consider the case of a storable good for which the exhaustion of the market fundamental requires the destruction of the good. This is for instance the case of gold. As an approximation, gold can at each period either be consumed or be held as a store of value, but not both, contrary to paintings, stocks, money, or land, for example. The problem caused by this feature is best exemplified using the metaphor of the black schmoos developed in Section 7. At each period the dividend associated with a black schmoo is zero; so is the market fundamental. However since the notion of market fundamental is implicitly linked with the "real value" of this asset, few people would say that the market fundamental of a black schmoo is

[29] Let x_t denote the quantity extracted at time t; X_t the total amount extracted by time t; V_t the total value on the extracted ore market; and W_t the value of stocks. By definition:

$$V_t = p_t X_t \quad \text{and} \quad W_t = \sum_{\tau=1}^{\infty} \frac{p_{t+\tau} x_{t+\tau}}{(1+r_{t+1}) \cdots (1+r_{t+\tau})} + e_t,$$

where $e_t = (1+r_t) \cdots (1+r_1) e_0$ is a bubble. Arbitrage on the ore market implies that $p_{t+1} = (1+r_{t+1}) p_t$. Straightforward computations show that $(V_{t+1} + W_{t+1}) = (1+r_{t+1})(V_t + W_t)$.

zero. They would rather assume it to be one, which is its consumption value. We are thus led to distinguish between the *financial market fundamental* (i.e., as seen by investors) (zero) and the *real market fundamental* (one). By taking the differences between the market price and these market fundamentals, one obtains the *financial bubble* and the *real bubble.* Properly speaking the real bubble is not a bubble, because it does not grow at the rate of interest. It rather is a measure of the overestimation of the asset relative to its real value. One should keep in mind that the (only) connection between the market price and the real market fundamental is that at each instant, the latter imposes a *lower bound* on the former.

(ii) Next I would like to mention the case of a firm that does not distribute dividends. This case is somewhat similar to the previous one. The financial market fundamental of the stock (equal to the present discounted value of dividends) is lower than the real market fundamental, which is equal to the present discounted value of profits minus the level of the current debt. And, as earlier, the real market fundamental serves as a lower bound on the market price.[30] It must therefore be the case that the stock of a firm that does not distribute dividends supports a financial bubble. This bubble again is unusual in that it is partially *backed* by the real market fundamental (or its excess over the present discounted value of dividends if the firm distributes some).

(iii) As a third example consider the economy described in Section 4, and assume that the reserve requirement on the backed intrinsically useless asset is small and non-binding along the equilibrium path ($\hat{b}_t > \xi \hat{s}_t$). The asset's market price grows at the rate of interest. Thus the asset is a pure bubble according to the financial definition. However if the asset's price were low, the consumers would not be able to meet the reserve requirement. In this sense the asset has a real value.

We thus conclude that the financial market fundamental may not be the right lower bound on an asset's market price. *If the price were to decrease towards its financial market fundamental, the asset would be used for other purposes not yet included in the definition of the market fundamental*: In (i), gold or black schmoos would be consumed; in (ii), the firm would be bought and possibly dividends would be paid; in (iii), the asset would be used to meet reserve requirements. We thus must take a flexible view of the market fundamental by distinguishing between the financial one (associated with arbitrage on the equilibrium path) and a "real one." A final caveat: There is sometimes ambiguity about the definition of the real market fundamental itself! Consider case (i) and assume that gold is essential and is used for consumption purposes (dentistry, for example). As we said the financial market fundamental of gold is zero. The real market fundamental can be defined as the current value of the stock of gold that is used for consumption. However this quantity, as well as the price, are endogeneous. One possibility is to take the value along the equilibrium path. Another possibility is to compute

[30] Otherwise there would be a take-over in our perfect information frictionless world. If the price is lower than the real value of the stock, some members of the current generation can buy the firm, distribute themselves a dividend equal to the real value by letting the firm borrow (the firm will then have a zero real value from this period on). These traders would thus make a profit.

the total value of gold in a fictitious economy in which bubbles on gold are ruled out (and therefore the whole stock of gold is consumed). These two values in general differ (see Tirole [**35**] for a comparison).

9. WHAT CREATES A BUBBLE?

In a sense I have been considering the demand for bubbles. The supply is virtually unlimited. For example I am always willing to pretend that a drawing I made when I was young is worth $1000, say. However I doubt I will be successful in convincing others that they should invest in it. If I were famous, I might be able to do so. Since almost anything is a potential source of bubble, how are actual bubbles, if any, selected?

There are three conditions that are necessary to creat a bubble: *Durability, scarcity,* and *common beliefs.* The common agreement requirement comes from the need for a focal point in the set of potential bubbles. The scarcity requirement stems from the fact that new units must have the same price as the old ones. The possibility of creating too much of this asset may prevent bubbles on this asset (see Wallace [**38**] and Tirole [**35**]).[31] The scarcity requirement explains why, at first sight, bubbles often affect assets that for historical reasons cannot be reproduced.[32]

10. CONCLUSION

To conclude I believe that the investigation of overlapping generations models should somewhat shift emphasis from the study of money to that of assets that are held for more speculative purposes. It is clear that the models and the empirical evidence are too preliminary to settle the question of whether we should expect to observe asset bubbles in overlapping generations economies. And we have not solved the old debate about which one of overlapping generations and infinitely lived consumers (or overlapping generations with bequests) is the "right" model. The difference between the two formalizations is substantial since bubbles may exist in the former but not in the latter. But I hope to have convinced the reader that in our current state of knowledge we would be best advised to believe that bubbles are not inconsistent with optimizing behavior and general equilibrium.

[31] It can be shown that the maximum rate of growth of an asset consistent with a bubble on this asset is the difference between the asymptotic rate of growth of the economy and the asymptotic rate of interest in the Diamond bubbleless economy. The possibility of future creation is indeed one of the reasons why Tobin [**37**] is somewhat sceptical about the use of overlapping generations models of money without transaction motives ("There is no governmental commitment to the value of money," p. 85).

[32] Potential examples are rare stamps, letters of past famous writers or personalities, famous paintings, gold, diamonds, land. For instance the market fundamental—consumption value—of rare stamps seems to be low—all the more that they generally sit in a bank; but their price can be very high. Artificial and real diamonds cannot be told apart with the naked eye. If there is no snobism effect, their market fundamental is basically the same, in spite of the fact that they command very different prices. If the real value of the asset depends on its price, the analysis is not as clear-cut; it then resembles that for money with transaction motives.

A good understanding of their definition and properties may be required in various fields such as empirical studies of asset pricing, monetary theory, and welfare economics.

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APPENDIX

APPENDIX 1

PROOF OF PROPOSITION 1 (Existence of Bubbles): (a) In a first step let us examine existence and uniqueness of a *bubbleless* equilibrium path. Let us first assume that $\bar{r}>0$.

LEMMA 1: *If* $\bar{r}>0$, *there exists a unique bubbleless equilibrium. The interest rate converges to* $\bar{r}$.

PROOF OF LEMMA 1: *Uniqueness*: Assume there are two paths corresponding to two different market fundamentals per capita $f_0'<f_0$ (but $w_0'=w_0$ and $r_o'=r_0$). Let us show by induction that $\forall t\geq 1$ $f_t'<f_t$ and $r_t'<r_t$. If these properties hold at t, then $w_t'>w_t$ and

$$r_{t+1}'=\psi(w_t',f_t')<r_{t+1}=\psi(w_t,f_t)$$

and

$$f_{t+1}'=\frac{1+r_{t+1}'}{1+n}f_t'-\frac{R}{(1+n)^{t+1}}<f_{t+1}=\frac{1+r_{t+1}}{1+n}f_t-\frac{R}{(1+n)^{t+1}}.$$

By definition of the market fundamentals, $r_0=r_0'$ and $\forall t\geq 1$ $r_t>r_t'$ imply that $f_0<f_0'$, a contradiction.

Existence: Define the function $f_0\to\Gamma(f_0)$ in the following way: Given f_0, equations (4), (5), (7), (9), and (10) define a path of interest rates. Let $\Gamma(f_0)$ be the present discounted value of the rent R at these interest rates. (If the path is not feasible, i.e., if for some t the capital stock becomes negative, then one takes $r_{t+\tau}=+\infty$, $\forall\tau\geq 0$.[33]) Clearly $\tilde{f}_0\equiv\Gamma(0)>0$; $\tilde{f}_0$ is the present discounted value of the rent computed with the Diamond bubbleless and rentless interest rates. From the uniqueness proof we know that the interest rates are always higher when the economy starts with a higher rent. Thus if one compares the economies starting with initial market fundamental 0 and initial market fundamental $\tilde{f}_0$, one has: $\Gamma(\tilde{f}_0)<\Gamma(0)$. But by definition $\Gamma(0)=\tilde{f}_0$. Thus $\Gamma(\tilde{f}_0)<\tilde{f}_0$. By continuity there exists f_0 such that $f_0=\Gamma(f_0)$. It is easy to see that the path starting with f_0 is feasible. For this path one has $\forall t$ $f_t=\Gamma(f_t)$. Imagine that at time t $s(w_t,r_t)\leq f_t$. Then $\Gamma(f_t)=0$. This clearly is impossible. Thus the path starting with f_0 is a perfect foresight path.

Convergence: Let us show that $\lim_{t\to\infty}f_t=0$. This is clearly the case if the interest rate is bounded away from 0 and positive. Assume there exists t such that $r_t<r_{t-1}$ and $r_t<n$. Then $w_t>w_{t-1}$ and

$$f_t=\frac{1+r_t}{1+n}f_{t-1}-\frac{R}{(1+n)^t}<f_{t-1}.$$

This in turn implies that $r_{t+1}=\psi(w_t,f_t)<r_t=\psi(w_{t-1},f_{t-1})$. By induction we obtain a sequence of ever decreasing interest rates and ever decreasing market fundamentals. Now from the uniqueness proof we know that the interest rate always exceeds that of the rentless economy. Thus the interest rate must in the long run exceed $(\bar{r}-\varepsilon)$ for $\varepsilon>0$. But then $F_t\leq R/(\bar{r}-\varepsilon)$ for t sufficiently large and therefore $\lim_{t\to\infty}f_t=0$ and $\lim_{t\to\infty}r_t=\bar{r}$. Thus we conclude that the economy behaves asymptotically like the Diamond rentless economy. *Q.E.D.*

Let us now show that if $\bar{r}<0$, there exists no equilibrium. Let us consider the three mutually exhaustive cases:

[33] $\Gamma(f_0)$ is well-defined; from the proof of uniqueness, $\Gamma(f_0)<\Gamma(0)$ and $\Gamma(0)<+\infty$ if $\bar{r}>0$.

(i) $\exists t$ such that $r_t < r_{t-1}$ and $r_t < n$. From the analysis in the convergence proof we know that $f_t/f_{t-1} < (1+r_t)/(1+n)$ and that interest rates are ever decreasing. Thus $\lim_{t\to\infty} f_t = 0$ and $\lim_{t\to\infty} r_t = \bar{r} < 0$. But this is impossible since F_t would be infinite.

(ii) $\forall t\ r_t \geqslant n$. Then $F_t \leqslant R/n$ and $\lim_{t\to\infty} f_t = 0$. Thus again $\lim_{t\to\infty} r_t = \bar{r} < 0$.

(iii) $\forall t\ r_t \geqslant r_{t-1}$. If there exists t such that $r_t > 0$, then $\forall \tau \geqslant 0\ F_{t+\tau} \leqslant R/r_t$ and therefore $\lim_{t\to\infty} f_t = 0$ and $\lim_{t\to\infty} r_t = \bar{r}$. If $\forall t\ r_t \leqslant 0$, then F_t is infinite and the young generation cannot buy the rent.

(b) Let us now consider *bubbles*. Let $a_t = b_t + f_t$ be the total amount of savings per capita that generation t invests in nonproductive activities. The behavior of the dynamical system is then determined by the following equations:

$$r_{t+1} = \psi(w_t, a_t), \tag{32}$$

$$w_t = \Phi(r_t), \tag{33}$$

$$a_{t+1} = \frac{1+r_{t+1}}{1+n} a_t - \frac{R}{(1+n)^{t+1}}. \tag{34}$$

Given $\{w_0, r_0\}$, the initial level of nonproductive savings a_0 determines a sequence $\{w_t, r_t, a_t\}$ as long as $s(w_t, r_{t+1}) > a_t > 0$. a_0 is then said to be *feasible*. It does not mean, however, that a_0 defines an equilibrium. The sequence of interest rates defines $f_t > 0$ using (9) and (10), and thus $b_t \equiv a_t - f_t$. An *equilibrium* a_0 is feasible and is such that $b_0 \geqslant 0$. The following lemmas are for feasible a_0, unless otherwise stated. For simplicity we rule out the nongeneric cases $\bar{r} = 0$ and $\bar{r} = n$.

LEMMA 2: *If there exists $t \geqslant 1$ such that $r_t < r_{t-1}$ and $r_t \leqslant n$, then (i) $0 < \bar{r} < n$ and (ii) the path is asymptotically bubbleless.*

PROOF OF LEMMA 2: If $r_t < r_{t-1}$, then $w_{t-1} = \Phi(r_{t-1}) < w_t = \Phi(r_t)$. On the other hand $r_t \leqslant n$ implies that $a_t < a_{t-1}$. These two inequalities in turn imply that $r_{t+1} = \psi(w_t, a_t) < r_t = \psi(w_{t-1}, a_{t-1})$. Thus by induction $n \geqslant r_t > r_{t+1} > \cdots > r_{t+\tau} > \cdots$. Therefore

$$b_{t+\tau} < b_{t+1}\left(\frac{1+r_{t+1}}{1+n}\right)^{\tau-1}$$

and the path is asymptotically bubbleless; and similarly for a: $\lim_{t\to\infty} a_t = 0$. Thus r_t converges to $\bar{r}$. As interest rates exceed those in the Diamond rentless and bubbleless equilibrium (by the same reasoning as in the proof of uniqueness in Lemma 1), $\bar{r} < n$. The interest rate sequence must converge to some positive value in order for f_t to be well-defined; hence $\bar{r} > 0$. *Q.E.D.*

LEMMA 3: *Either the interest rate converges to $\bar{r}$ and the economy is asymptotically bubbleless or the interest rate converges to n and the asymptotic bubble per capita is $\hat{b}$.*

PROOF OF LEMMA 3: Assume first that $\forall t\ r_t \geqslant r_{t-1}$. Then $\forall t\ r_t \leqslant n$, otherwise the bubble per capita would explode. If the interest rate converges to $r^* < n$, then the economy is asymptotically bubbleless. Moreover F_t decreases since the interest rate always increases. Therefore $\lim_{t\to\infty} f_t = 0$. Thus $\lim_{t\to\infty} a_t = 0$ and $r^* = \bar{r}$. Note that in order for F_t to be well-defined, $\bar{r}$ must be strictly positive. If the interest rate converges to n, then $\lim_{t\to\infty} f_t = 0$ and hence b_t must converge to $\hat{b}$.

Second assume that there exists t such that $r_t < r_{t-1}$. If $r_t < n$, we know from Lemma 2 that the economy is asymptotically bubbleless and that r_t converges to $\bar{r}$. Hence assume that $\forall \tau > 0$, $r_{t+\tau} > n$. The interest rate must then converge to n in order for the bubble per capita not to explode. And $\lim_{t\to\infty} f_t = 0$. As $r_{t+1} = \psi(r_t, a_t)$ and $n = \psi(\Phi(n), \hat{b})$, a_t, and thus b_t, must converge to $\hat{b}$. *Q.E.D.*

LEMMA 4: *Assume $\bar{r} > 0$. For a given initial capital stock, if the path is feasible for initial a_0, the path determined by $a_0' < a_0$ is also feasible. Furthermore if* $\lim_{t\to\infty} r_t = \bar{r}$, *then* $\lim_{t\to\infty} r_t' = \bar{r}$.

PROOF OF LEMMA 4: We know that $r_1' < r_1$. We show by induction that $\forall t \geqslant 1$, $r_t' < r_t$ and $a_t' < a_t$. If these hold at time t, then $w_t' > w_t$ and

$$a_{t+1}' = \frac{1+r_{t+1}'}{1+n} a_t' - \frac{R}{(1+n)^{t+1}} < a_{t+1} = \frac{1+r_{t+1}}{1+n} a_t - \frac{R}{(1+n)^{t+1}}.$$

Moreover $r'_{t+1}=\psi(w'_t, a'_t)<r_{t+1}=\psi(w_t, a_t)$. Thus there is always more capital accumulation in the path starting with a'_0. Lastly if $r_t \to \bar{r}$, then $a_t \to 0$ and therefore r'_t can only converge to $\bar{r}$ (from Lemma 3). *Q.E.D.*

LEMMA 5: *If $\bar{r}<n$, there exists at most one value of a_0 such that the interest rate converges to n.*

PROOF OF LEMMA 5: Consider $a'_0<a_0$. If for both values the interest rate converges to n, the economies are asymptotically rentless (per capita) and from Lemma 3, $\lim_{t\to\infty} b'_t=\lim_{t\to\infty} a'_t=\lim_{t\to\infty} b_t=\lim_{t\to\infty} a_t=\hat{b}$. But from the proof of Lemma 4 we know that $r'_t<r_t$ and $a'_t<a_t\ \forall t\geq 1$. Hence:

$$\frac{a'_t}{a_t}=\frac{\dfrac{1+r'_t}{1+n}a'_{t-1}-\dfrac{R}{(1+n)^t}}{\dfrac{1+r_t}{1+n}a_{t-1}-\dfrac{R}{(1+n)^t}}<\frac{(1+r'_t)a'_{t-1}}{(1+r_t)a_{t-1}}<\frac{a'_{t-1}}{a_{t-1}}<\cdots<\frac{a'_0}{a_0}<1.$$

Thus a'_t/a_t does not converge to 1. *Q.E.D.*

LEMMA 6: *If $\bar{r}<n$, the set of equilibrium a_0 is convex. Furthermore $db_0/da_0>1$ and $df_0/da_0<0$.*

PROOF OF LEMMA 6: For a given a_0 and from Lemma 3, either r_t converges to $\bar{r}$ or n or the path is not feasible because at some point of time a_t exceeds the total amount of savings per capita. Let a'_0 and $a''_0(a'_0<a''_0)$ denote two equilibrium initial nonproductive savings levels; and consider $\tilde{a}_0\in(a'_0, a''_0)$. First from the proof of Lemma 4 and with obvious notation $\forall t\ a'_t<\tilde{a}_t<a''_t$ and $r'_t<\tilde{r}_t<r''_t$. This shows that at any t nonproductive savings are lower than total savings with initial $\tilde{a}_0$ since they are for initial a''_0. Thus the path starting with initial nonproductive savings $\tilde{a}_0$ is feasible. Let $f_0(a_0)$ denote the market fundamental of rents computed with the sequence of interest rates associated with a_0. And let $b_0(a_0)\equiv a_0-f_0(a_0)$. The path is an equilibrium if and only if $b_0(\tilde{a}_0)\geq 0$. Let us finally notice that $db_0/da_0>1$ and $df_0/da_0<0$: From the proof of Lemma 4, we know that $a'_0<a_0$ implies $\forall t\geq 1\ r_t>r'_t$. Therefore $f'_0<f_0$. Thus $df_0/da_0<0$. But $db_0/da_0=1-(df_0/da_0)>1$.

Since $b_0(a'_0)\geq 0$, we conclude that $b_0(\tilde{a}_0)>0$ and therefore the path starting with $\tilde{a}_0$ is an equilibrium. *Q.E.D.*

LEMMA 7: *If $0<\bar{r}<n$, then any "sufficiently small" initial bubble is an equilibrium bubble.*

PROOF OF LEMMA 7: From Lemma 1 we know that in the bubbleless equilibrium the interest rate converges to $\bar{r}$ and the economy is asymptotically rentless in per capita terms. Since ψ and Φ are continuous, for any $\varepsilon>0$, there exists T sufficiently large and b_0 sufficiently small such that $r_T<\bar{r}+\varepsilon$ and $a_T=f_T+b_T<\alpha$ where $\alpha>0$ is defined by: $\psi(\Phi(\bar{r}+\varepsilon), \alpha)=\bar{r}+\varepsilon$. Let us show that, $\forall\tau\geq 0$, $r_{T+\tau}<\bar{r}+\varepsilon$ (which implies that the path defined by b_0 is feasible). We know that: $w_T=\Phi(r_T)>\Phi(\bar{r}+\varepsilon)$ and $a_T<\alpha$; thus:

$$r_{T+1}=\psi(\Phi(r_T), a_T)<\psi(\Phi(\bar{r}+\varepsilon), \alpha)=\bar{r}+\varepsilon.$$

Now if ε is chosen so that $\bar{r}+\varepsilon<n$, $a_{T+1}<a_T<\alpha$. The claim is then proved by induction. *Q.E.D.*

LEMMA 8: *If $\bar{r}>n$, no bubble is sustainable.*

PROOF OF LEMMA 8: From the proof of Lemma 4, the interest rate at each time is higher than in the bubbleless and rentless economy. Thus for t sufficiently large $r_t>\bar{r}-\varepsilon>n$. The bubble per capita then grows exponentially and eventually exceeds savings per capita, which is impossible. *Q.E.D.*

LEMMA 9: *The set of feasible a_0 such that r_t converges to $\bar{r}$ is open on the right.*

PROOF OF LEMMA 9: Assume that a_0 is such that r_t converges to $\bar{r}$. Let us show that for $a'_0=a_0+\eta$ and η positive, sufficiently small, the interest rate r'_t converges to $\bar{r}$. As in the proof of Lemma 7, for any $\varepsilon>0$, there exists T sufficiently large such that $r_T<\bar{r}+\varepsilon$ and $a_T<\alpha$ where $\alpha>0$ is defined by

$\psi(\Phi(\bar{r}+\varepsilon), \alpha) = \bar{r}+\varepsilon$. By continuity of ψ and Φ, *and for* η sufficiently small, $r'_T < \bar{r}+\varepsilon$ and $a'_T < \alpha$. By the same proof as in Lemma 7, r'_t also converges to $\bar{r}$. *Q.E.D.*

LEMMA 10: *The set of equilibrium a_0 is closed on the right. And similarly for the set of feasible a_0.*

PROOF OF LEMMA 10: Let $\{a_0^n\}$ denote an increasing equilibrium sequence converging to $\hat{a}_0$; and let $\{r_t^n, w_t^n\}$ denote the corresponding sequences of interest rates and wages. First let us show that for all t, the sequence $\{s(w_t^n, r_{t+1}^n) - a_t^n\}$ is bounded away from zero (note that from the proof of Lemma 4 and equation (11) this sequence is decreasing). If not, for some t, $\lim_{n\to\infty}\{s(w_t^n, r_{t+1}^n) - a_t^n\} = 0$ and thus $\lim_{n\to\infty}\{r_{t+1}^n\} = +\infty$. Now remembering that for any τ, $r_\tau^n > \bar{r}_\tau$, where $\bar{r}_\tau$ denotes the interest rate in the Diamond bubbleless and rentless equilibrium,

$$b_{t+1}^n \geq b_0^0 \frac{(1+\bar{r}_1)\cdots(1+\bar{r}_t)}{(1+n)^t}(1+r_{t+1}^n).$$

Thus, fixing t, $\lim_{n\to\infty} b_{t+1}^n = +\infty$ (since $db_0/da_0 > 0$, b_0^0 can be chosen strictly positive). But we know that $\forall n$, $w_{t+1}^n < \bar{w}_{t+1}$, where $\bar{w}_{t+1}$ denotes the wage in the Diamond bubbleless and rentless equilibrium. Hence for n sufficiently large the bubble at time $(t+1)$ exceeds income and therefore savings. This means that the corresponding path is not feasible, a contradiction. Lastly define

$$\hat{r}_t \equiv \lim_{n\to\infty}\{r_t^n\}, \qquad \hat{w}_t \equiv \lim_{n\to\infty}\{w_t^n\}, \qquad \hat{a}_t \equiv \lim_{n\to\infty}\{a_t^n\}.$$

The path $\{\hat{r}_t, \hat{w}_t, \hat{a}_t\}$ satisfies (4), (7), (10), (11), (12), and (15), i.e., is an equilibrium. The proof is similar for the set of feasible a_0 (a_t^n goes to $+\infty$ for some t). *Q.E.D.*

Let us now conclude the proof. First assume that $0 < \bar{r} < n$. Lemma 1 tells us that there exists an $\bar{a}_0$ that leads to a bubbleless equilibrium. Lemmas 6 and 7 ensure that the equilibrium set is convex, has a nonempty interior, and is bounded below by $\bar{a}_0$. Let $\hat{a}_0$ denote the supremum in this set ($\hat{a}_0 < w_0$). From Lemma 10, $\hat{a}_0$ leads to an equilibrium. From Lemmas 3, 5, and 9, for $a_0 \in [\bar{a}_0, \hat{a}_0)$ ($a_0 = \hat{a}_0$), the interest rate converges to $\bar{r}(n)$; and $a_0 \notin [\bar{a}_0, \hat{a}_0]$ is not an equilibrium:

The functions $f_0(a_0)$ and $b_0(a_0)$ are represented in Figure 3 ($a_0 \in [\bar{a}_0, \hat{a}_0) \Rightarrow r_t \to \bar{r}$; $a_0 = \hat{a}_0 \Rightarrow r_t \to n$).

Next assume that $\bar{r} < 0$. From Lemma 3, for a given a_0, the interest rate converges to $\bar{r}$ or to n or the path is not feasible. If the interest rate converges to $\bar{r}$, the market fundamental $f_0(a_0)$ is infinite and a_0 is not an equilibrium. Lemma 9 and 10 show that there exists $\hat{a}_0$ such that the corresponding path is feasible and the interest rate converges to n. The last step consists in checking that this path is an equilibrium path, i.e., that $b_0(\hat{a}_0) \geq 0$. Imagine that $b_0(\hat{a}_0) < 0$. Then the bubble is always negative. As the interest rate converges to n, F_t converges to (R/n) and f_t converges to 0. Thus a_t becomes arbitrarily small, or negative, and the interest rate cannot converge to n from (8). *Q.E.D.*

The last two lemmas check that, in the asymptotically bubbly path, nonproductive savings increase with the current level of capital; and that, if there are no rents, convergence to the golden rule is monotonic.

LEMMA 11: $d\hat{a}_0/dk_0 > 0$.

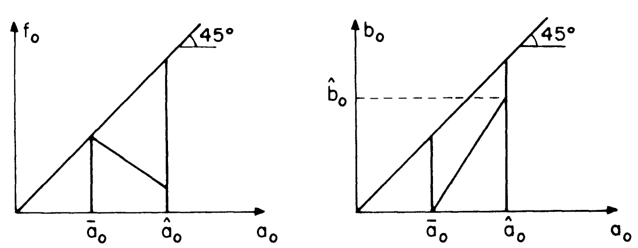

FIGURE 3.

PROOF OF LEMMA 11: Imagine that $k_0 < k_0'$ and $\hat{a}_0 > \hat{a}_0'$ (with obvious notation). Then, as in the proof of Lemma 4, $\forall t\ \hat{r}_t > \hat{r}_t'$. Therefore

$$\forall t \quad \frac{\hat{a}_t}{\hat{a}_t'} > \frac{(1+\hat{r}_t)}{(1+\hat{r}_t')}\frac{\hat{a}_{t-1}}{\hat{a}_{t-1}'} > \cdots > \frac{\hat{a}_0}{\hat{a}_0'} > 1.$$

As $\hat{a}_t$ and $\hat{a}_t'$ must both converge to $\hat{b}$, we obtain a contradiction. *Q.E.D.*

LEMMA 12: *In the no-rent case, the asymptotically bubbly equilibrium converges monotonically* (*to the golden rule*).

PROOF OF LEMMA 12: Let $(\hat{r}_t, \hat{w}_t, \hat{k}_t, \hat{b}_t)$ denote the values of the variables on the asymptotically bubbly path. First assume that $\hat{r}_t > n$. Let us show that $n < \hat{r}_{t+1} < \hat{r}_t$. From Lemma 2, we know that $\hat{r}_{t+1} \geq n$. Imagine that $\hat{r}_{t+1} \geq \hat{r}_t$, i.e., that $\hat{k}_{t+1} \leq \hat{k}_t$. From Lemma 11 (with $\hat{a} = \hat{b}$, since $R = 0$), we have $\hat{b}_{t+1} \leq \hat{b}_t$, in order to remain on the stable manifold. But $\hat{b}_{t+1} > \hat{b}_t$, as $\hat{r}_{t+1}$ exceeds n, a contradiction. Second assume that $\hat{r}_t < n$. From Lemma 2, we know that $\hat{r}_{t+1} > \hat{r}_t$. Can one have $\hat{r}_{t+1} \geq n$? Then $\hat{r}_{t+1} = \psi(\hat{w}_t, \hat{b}_t) < \hat{r}_{t+2} = \psi(\hat{w}_{t+1}, \hat{b}_{t+1})$ since $\hat{b}_{t+1} \geq \hat{b}_t$ and $\hat{w}_{t+1} < \hat{w}_t$. By induction one obtains an increasing sequence of interest rates above n; so the bubble explodes, a contradiction. *Q.E.D.*

PROOF OF PROPOSITION 2: If the economy is asymptotically bubbleless, then

$$\lim_{t\to\infty} \frac{(1+r_1)\cdots(1+r_t)}{(1+n)^t} = 0.$$

On the other hand if the economy is asymptotically bubbly,

$$\lim_{t\to\infty} \frac{(1+r_1)\cdots(1+r_t)}{(1+n)^t} = \lim_{t\to\infty} \frac{\hat{b}_t}{\hat{b}_0} = \frac{\hat{b}}{\hat{b}_0} > 0.$$

By (a straightforward extension of) Theorem 5.6 in Balasko-Shell [3], the asymptotically bubbleless equilibria are inefficient and the asymptotically bubbly one is efficient. *Q.E.D.*

PROOF OF PROPOSITION 7 (No Monetary Bubble in the Money-in-Utility-Function Model): Let us assume that money is not backed through a reserve requirement. Its market fundamental is then:

$$f_0 = \sum_{t\geq 0} \frac{p_t}{(1+r_1)\cdots(1+r_t)} \frac{u_m^t}{u_y^t}. \tag{35}$$

Now, if there is a bubble e_0 on money and money is not backed,

$$p_t \geq e_0(1+r_1)\cdots(1+r_t).$$

Therefore a necessary condition for a bubble on money to exist is that

$$\sum_{t\geq 0} \frac{u_m^t}{u_y^t} < +\infty.$$

In particular it must be the case that

$$\lim_{t\to\infty} \frac{u_m^t}{u_y^t} = 0.$$

But if $(1-\varepsilon)$ is an upper bound on the propensity to save,

$$\frac{m_t}{c_t^y} = \frac{\dfrac{p_t M_t}{(1+n)^t}}{c_t^y} \leq \frac{s_t}{c_t^y} \leq \frac{1-\varepsilon}{\varepsilon}.$$

Using Assumption A, $\{u_m^t/u_y^t\}$ does not converge to zero, a contradiction. *Q.E.D.*

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