2014/2015, week 3

The Solow Growth Model

Romer, Chapter 1.1 to 1.5

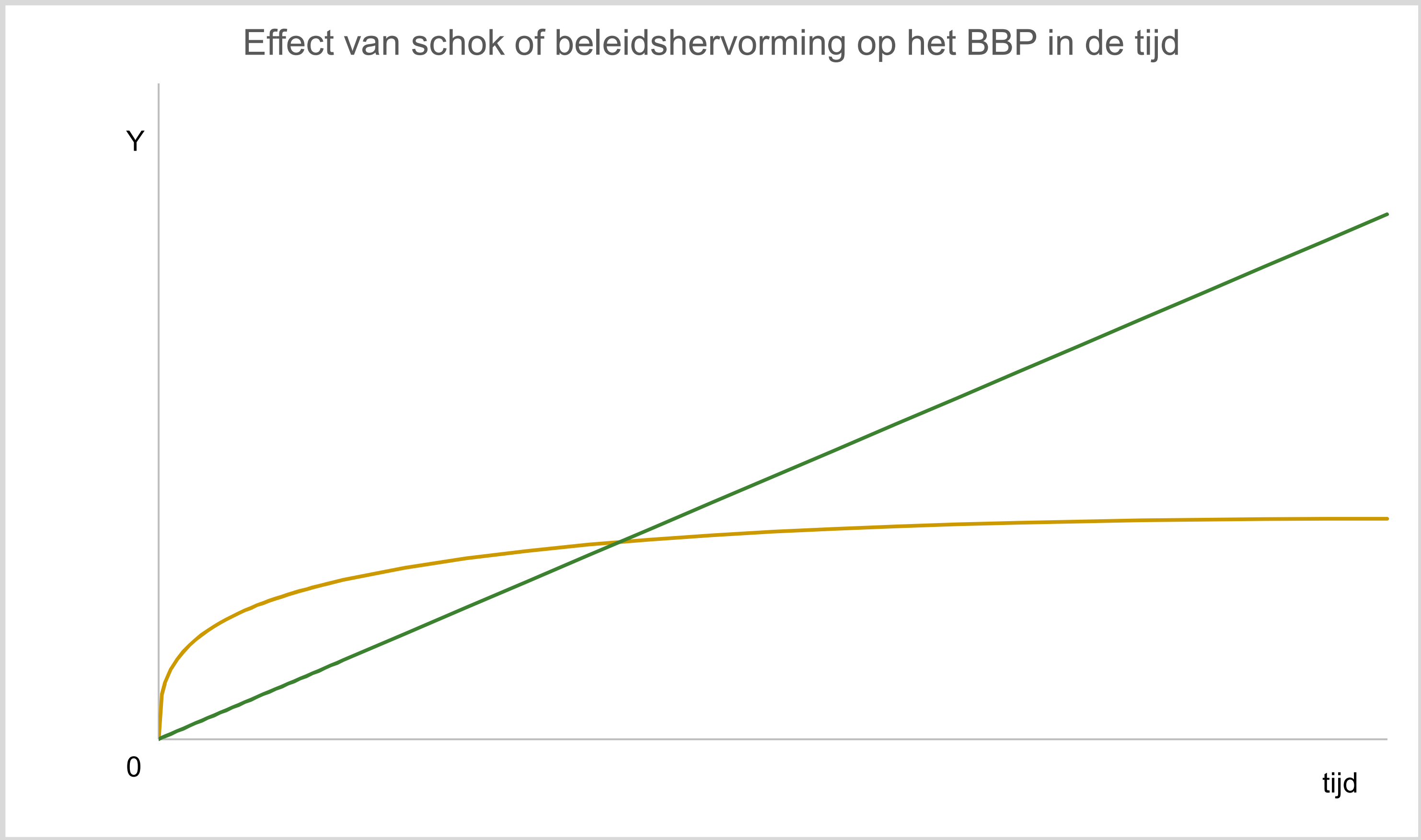
Facts

- Economic growth is about how economies evolve over time
- Economic growth then refers to the growth of the Gross Domestic Product (GDP) (in real terms)
- Economists rather talk about GDP per capita (in real terms)
- GDP ≠ Welfare
  - Home work
  - Environment
  - Natural resources

Facts

- GDP
  - Gross rather than net
  - Domestic rather than National
  - Product = Expenditure = Income
- Do not confuse economic growth and a high level of GDP
  - High growth rates for a period of time do not necessarily translate into high standard of living
  - For example, if Bangladesh achieves a 5 percent rate of growth, it will take 60 years to reach the current level of real income in the US

Level and Growth Effects

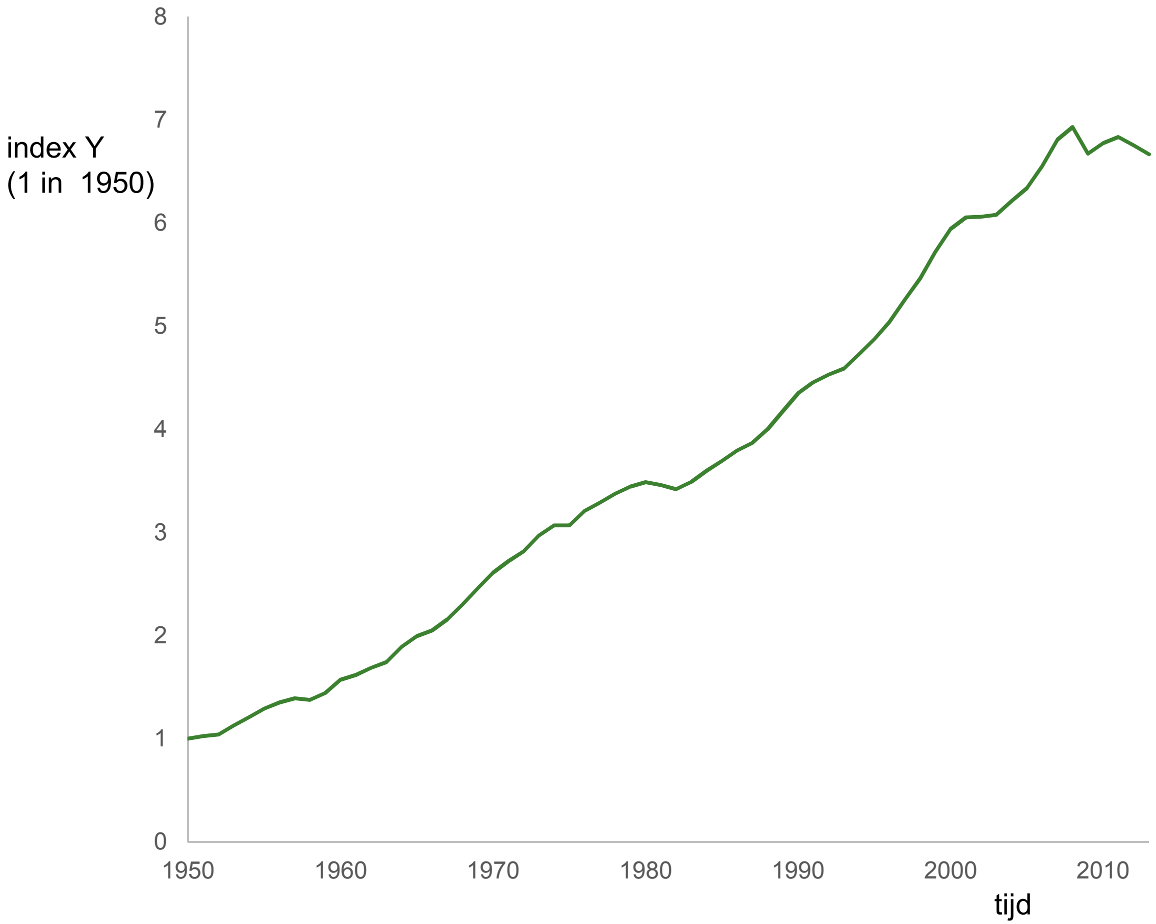


- Bron: *Economen kunnen niet rekenen*

Facts

- Facts about economic growth
  - Since the industrial revolution, almost constant growth, interrupted by crises
  - Growth rates differ over time
  - Growth rates differ between countries

GDP, The Netherlands



- Bron: *Economen kunnen niet rekenen*

Facts

- Average growth rates in industrialized countries were higher in the 20th century than in the 19th, and higher in the 19th than in the 18th
- Before the industrial revolution, probably no or insignificant growth

Growth rates differ between countries

- Growth miracles: Japan, South Korea, Taiwan, Singapore and Hong Kong from the end of WW II to 1990
- Average incomes in the above countries have grown at an average annual rate of over 5% since 1960
- Growth disasters: a country's growth lies well below the world average
- For example, Argentina was a miracle in 1900: income per capita close to that of industrialized countries
  - During the 20th century, growth in Argentina was bad
  - Currently, the level of income per capita in Argentina is close to the world average

Facts

Tabel 10.2: Verdeling van inkomen en economische groei in geïndustrialiseerde landen
Bron: Heston *et al.* (2011)

	BBP per hoofd van de bevolking, 1970 (in $)	BBP per hoofd van de bevolking, 2009 (in $)	Economische groei per jaar, 1970-2009 (in %)
VS	20.480	41.102	1,8
Nederland	19.050	40.566	2,0
Duitsland	16.236	32.487	1,8
Verenigd Koninkrijk	15.829	33.386	1,9
Frankrijk	15.676	30.821	1,7
Italië	14.371	27.692	1,7
Spanje	11.981	27.632	2,2
Zuid-Korea	3.018	25.029	5,6

Facts

Tabel 10.3: Verdeling van inkomen en economische groei in de wereld
Bron: Heston *et al.* (2011)

	BBP per hoofd van de bevolking, 1970 (in $)	BBP per hoofd van de bevolking, 2009 (in $)	Economische groei per jaar, 1970-2009 (in %)
VS	20.480	41.102	1,8
Nederland	19.050	40.566	2,0
Venezuela	8.934	9.115	0,1
Madagascar	950	753	-0,6
India	886	3.238	3,4
China	865	7.431	5,7
Oeganda	817	1.152	0,9
Zimbabwe	339	143	-2,2

The Solow-model of economic growth

- Stylized facts (empirical observations) by Nicholas Kaldor (1957):
  - Rate of return on capital, capital-output ratio and shares of capital income and labour income in national income roughly constant over time
  - Output-labour ratio, capital-labour ratio and wage rate exhibit almost constant growth over time

Assumptions of the Solow-model

- Closed economy
  - No foreign trade
  - Saving equal to investment
  - Endogenous interest rate
- Aggregate production function
- Savings rate
  - Exogenous
- Capital accumulation equation

Aggregate production function

- $Y(t) = F\big(K(t), A(t)L(t)\big)$
  - Y denotes output
  - K denotes the capital stock
  - A is an index of labour productivity
  - L denotes labour
  - t denotes time
  - AL is called effective labour
- Capital, labour and knowledge A are the production factors
- In this specification, technology (or knowledge) is labour-augmenting (or Harrod-neutral). That is, technological innovations (e.g. increases in A) come through labour input

Aggregate production function

- An alternative concept of technological change is that A works on capital:
  - $Y(t) = F\big(A(t)K(t), L(t)\big)$
- A second alternative concept of technological change is that A works on capital and labour:
  - $Y(t) = A(t)F\big(K(t), L(t)\big)$
  - This is called Hicks-neutral technological change
- In the Solow-model, technological change is of the Harrod-neutral type

Assumptions about the production function

- Constant returns to scale (CRS):
  - Increasing the amount of all inputs by a constant c increases output by the same factor c. This is called homogeneity of degree one in maths
  - Formally, $F\big(cK(t), cA(t)L(t)\big) = \mathrm{c}F(K(t), A(t)L(t))$

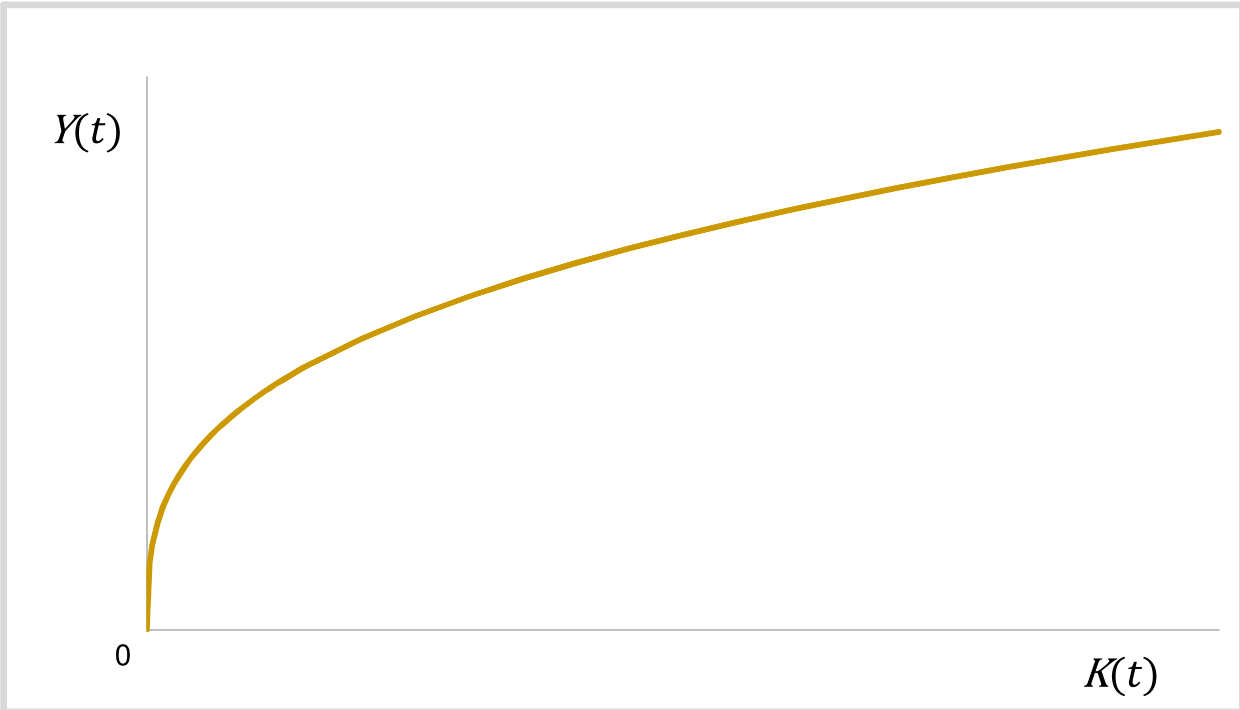
- Cobb-Douglas production function special case:
  - $F\big(K(t), A(t)L(t)\big) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$
  - $F\big(cK(t), cA(t)L(t)\big) = (cK(t))^{\alpha}(cA(t)L(t))^{1-\alpha} = c(K(t))^{\alpha}(A(t)L(t))^{1-\alpha} = cF\big(K(t), A(t)L(t)\big)$

Assumptions about the production function

- Implication of CRS property is that we can write down the production function in intensive form:
  - $y(t) \equiv \frac{Y(t)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) = f\big(k(t)\big)$
  - where $y(t)$ measures output per effective labour and $k(t)$ measures output per effective labour
- Cobb-Douglas production function:
  - $y(t) \equiv \frac{Y(t)}{A(t)L(t)} = \frac{K(t)^{\alpha}\big(A(t)L(t)\big)^{1-\alpha}}{A(t)L(t)} = \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} = k(t)^{\alpha}$

Further assumptions about the production function

- Declining marginal product of capital
  - $F_K > 0, F_{KK} < 0$



Further assumptions about the production function

- Declining marginal product of capital
  - $F_K > 0, F_{KK} < 0 \ \rightarrow f_k > 0, f_{kk} < 0$
- Inada conditions:
  - $\lim_{k \to 0} F_K = \infty$
  - $\lim_{k \to \infty} F_K = 0$
- Intuition:
  - As the stock of capital per effective labour approaches zero, its marginal product becomes larger and larger
  - As the stock of capital per effective labour gets increasingly larger, its marginal product will fall to almost zero

Cobb-Douglas production function

- $F\big(K(t), A(t)L(t)\big) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$

- Intensive form:
  - $f(k_t) = k(t)^{\alpha}$

- Properties:
  - $f'(k_t) = \alpha k(t)^{\alpha-1} > 0$
  - $f''(k_t) = \alpha(\alpha - 1)k(t)^{\alpha-2} < 0$

Solow model

- Closed economy:
  - $I = S$, where I denotes investment and S denotes savings
- Capital cannot adjust instantaneously. Rather, capital accumulates slowly over time. The process is described by the capital accumulation equation:
  - $\dot{K}(t) = I(t) - \delta K(t)$
  - where $\dot{K}(t) = dK(t)/dt$, the derivative of K with respect to time

Further assumptions

- Labour grows at a constant rate:
  - $\dot{L}(t)/L(t) = n$
- Labour-augmenting technological change is also a constant:
  - $\dot{A}(t)/A(t) = g$

Further assumptions

- The saving rate determines the allocation of output or income over consumption and savings
- The saving rate is exogenous
- The Solow-model is a Keynesian-type of model rather than a classical type of model

The accumulation equation for k

- Recall $k(t) \equiv K(t)/(A(t)L(t))$
- Apply the chain rule to $k(t)$:
  - $\dot{k}(t) = \frac{\partial k(t)}{\partial K(t)}\dot{K}(t) + \frac{\partial k(t)}{\partial A(t)}\dot{A}(t) + \frac{\partial k(t)}{\partial L(t)}\dot{L}(t)$
- Hence, applying the chain rule, we derive:
  - $\dot{k}(t) = \frac{1}{A(t)L(t)}\dot{K}(t) - \frac{K(t)L(t)}{\left(A(t)L(t)\right)^2}\dot{A}(t) - \frac{K(t)A(t)}{\left(A(t)L(t)\right)^2}\dot{L}(t)$
  - $= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{\left(A(t)L(t)\right)^2}\left(L(t)\dot{A}(t) + A(t)\dot{L}(t)\right)$
  - $= \frac{\dot{K}(t)}{A(t)L(t)} - k(t)\left(\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right)$

The accumulation equation for *k*

- $\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - k(t)\left(\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right)$

- Recall
  - $\dot{K}(t) = I(t) - \delta K(t) \ = sY(t) - \delta K(t)$
  - $\dot{A}(t)/A(t) = g$
  - $\frac{\dot{L}(t)}{L(t)} = n$

- Combined:
  - $\dot{k}(t) = \frac{sY\ (t) - \delta K(t)}{A(t)L(t)} - k(t)(g + n)$
  - $\dot{k}(t\ ) = sy(t) - k(t)(\delta + n + g)$

The accumulation equation for k

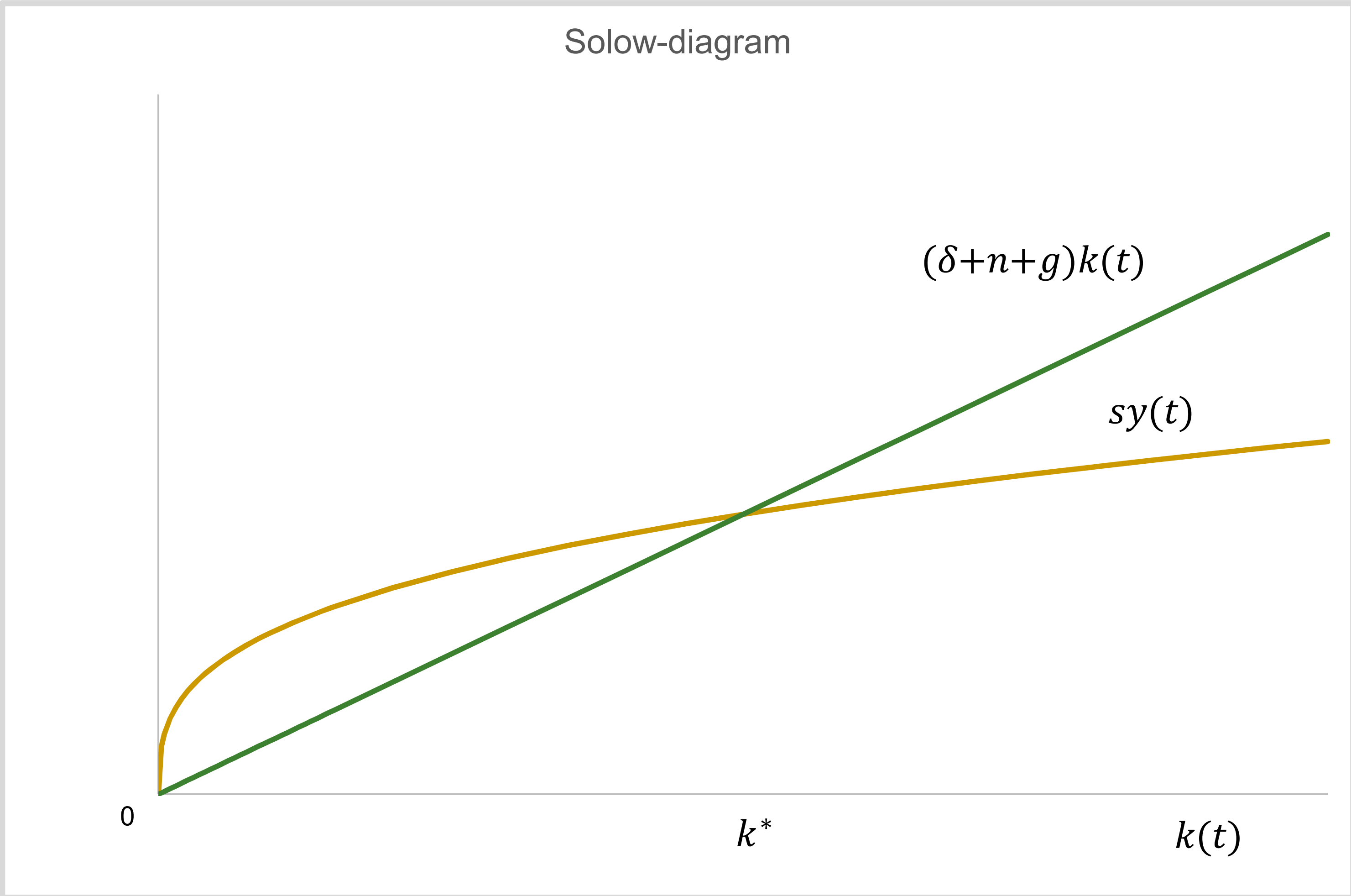
- $\dot{k}(t) = sy(t) - k(t)(\delta + n + g)$

- Break-even investment, $k(t)(\delta + n + g)$, is that amount of investment that keeps capital per effective labour constant over time
- Capital per effective labour thus increases if investment, $sy(t)$, exceeds break-even investment

The accumulation equation for k

- k^* refers to the steady-state value of k
- Hence, by definition, $\dot{k}^* = 0$
- This is the point of intersection of the investment curve and the break-even investment line in the Solow-diagram

The accumulation equation for k



The accumulation equation for k

- This means, for $k(t) < k^*$, $k(t)$ will increase over time
- This continues until $k(t)$ reaches k^*

- For $k(t) > k^*$, $k(t)$ will decrease over time
- This also continues until $k(t)$ reaches k^*

- Combined, we derive that the steady-state equilibrium is stable

The steady state

- $y(t) \equiv \frac{Y(t)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) = f\big(k(t)\big)$

- Capital per effective labour features zero growth over time in steady state: $\dot{k}^*(t)/k^*(t) = 0$

- What about other variables?
  - $K(t) \equiv k(t)A(t)L(t)$
  - $\frac{\dot{K}^*(t)}{K^*(t)} = \frac{\dot{k}^*(t)}{k^*(t)} + \frac{\dot{A}^*(t)}{A^*(t)} + \frac{\dot{L}^*(t)}{L^*(t)} = 0 + g + n = g + n$

The steady state

- Similarly,
  - $y(t) = f(k(t))$
  - $\frac{\dot{y}^*}{y^*} = \frac{f'(k^*)\dot{k}^*}{y^*} = 0$
  - $Y(t) \equiv y(t)A(t)L(t)$
  - $\dot{Y}^*/Y^* = \frac{\dot{y}^*}{y^*} + \frac{\dot{A}^*}{A^*} + \frac{\dot{L}^*}{L^*} = 0 + \mathrm{g} + n = g + n$

Kaldor's stylized facts

- The steady state of the Solow model thus conforms to the stylized facts as observed by Kaldor

- In the steady-state equilibrium of the Solow-model:

  - Rate of return on capital (interest rate), capital-output ratio and shares of capital income and labour income in national income are constant

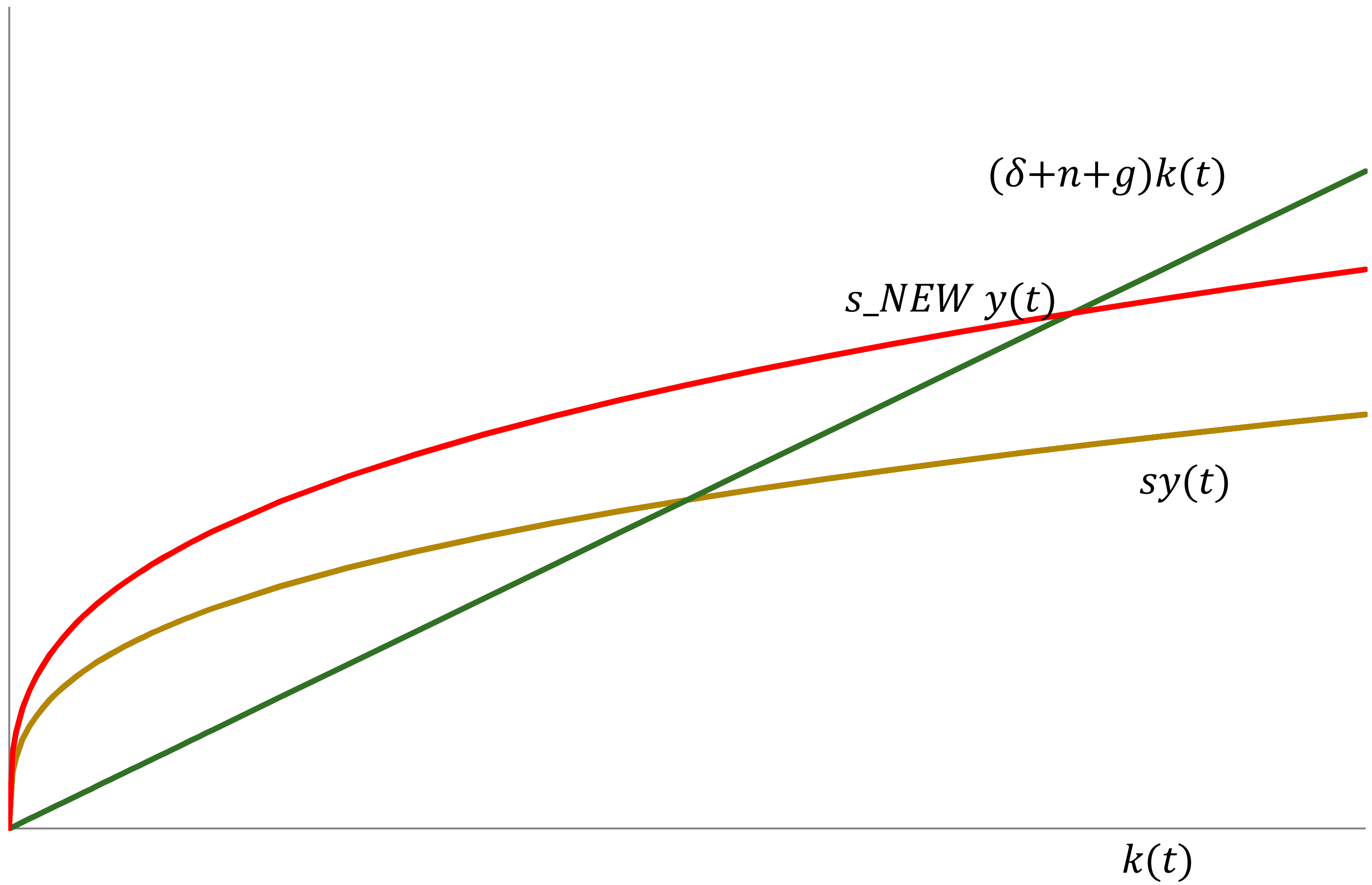
  - Output-labour ratio, capital-labour ratio and wage rate grow at rate g over time

Effects of a higher s or δ

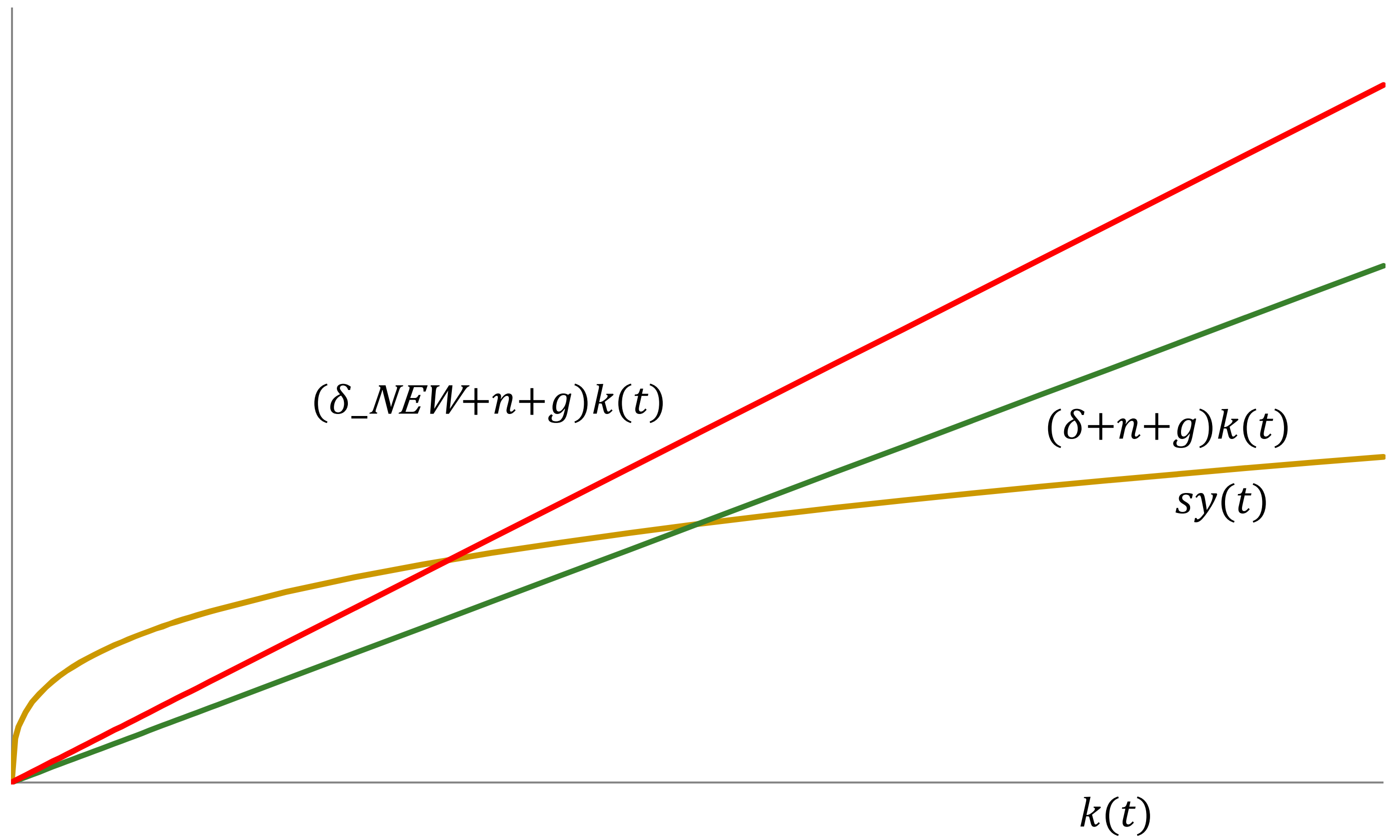
- A permanent increase in the saving rate (s) increases the steady-state value of capital per effective labour

- A permanent increase in the rate of depreciation of capital (δ), the rate of population growth (n) or the rate of technological progress (g) decreases the steady-state value of capital per effective labour

The effects of an increase in the saving rate



The effects of an increase in the rate of depreciation



Effects of a higher s or δ

- A permanent increase in s increases k^*
- A permanent increase in δ decreases k^*

- Cobb-Douglas specification: $y(t) = k(t)^\alpha$

- Steady-state equilibrium:
  - $\dot{k}(t) = sy(t) - k(t)(\delta + n + g) = 0 \quad \rightarrow$
  - $sk(t)^\alpha = k(t)(\delta + n + g)$

Effects of a higher s or δ

- Combine:
  - $s(k^*)^\alpha = k^*(\delta + n + g)$ $\rightarrow$
  - $(k^*)^{1-\alpha} = \frac{s}{\delta+n+g}$ $\rightarrow$
  - $k^* = \left(\frac{s}{\delta+n+g}\right)^{1/(1-\alpha)}$
- An increase in s (δ) increases (decreases) k^*
  - $y^* = \left(\frac{s}{\delta+n+g}\right)^{\alpha/(1-\alpha)}$
- An increase in s (δ) increases (decreases) y^*

Level and growth effects

- These effects are so-called level effects:
  - Ultimately, the shocks increase or decrease the level of the variable, but not its growth rate
  - Hence, in a new steady-state equilibrium, variables grow at their initial rate, but at a lower (or higher) level

Effects of a higher *n* or *g*

- A permanent increase in the rate of population growth (*n*) or the rate of technological progress (*g*) decrease k^*
- The effects combine level and growth effects:
  - The shocks increase or decrease the level of the variable, according to the Solow diagram
  - The growth rate of variables may change as well on account of changes in *n* or *g*

The golden rule of capital accumulation

- An increase in the saving rate increases the steady-state level of capital per effective labour and the steady-state level of output per effective labour
- Does this increase steady-state consumption per effective labour as well?
- Steady-state consumption per effective labour reads as:
  - $c^* = (1 - s)f(k^*)$
- An increase in s increases c^* by increasing $f(k^*)$ and decreases it through the term $(1 - s)$. Hence, the effect of an increase in s upon c^*depends on which of the two effects dominates

The golden rule of capital accumulation

- The fact that we have a declining marginal product of capital means that the positive effect upon c^* diminishes when k^*gets larger

- This suggests that there is a value of k^* for which c^* is maximal. Indeed, this value of k^* exists; it is called the golden role of capital accumulation

  - $c^* = (1-s)f(k^*) = f(k^*) - (\delta + n + g)k^*$

  - $\frac{\partial c^*}{\partial k^*} = f'(k^*) \ - (\delta + n + g) = 0$

The golden rule of capital accumulation

- $\frac{\partial c^*}{\partial {k^*}_{GR}} = f'({k^*}_{GR}) - (\delta + n + g) = 0$

- Cobb-Douglas production function:
  - $f(k) = k^{\alpha} \rightarrow f'(k) = \alpha k^{\alpha-1}$
  - $\alpha({k^*}_{GR})^{\alpha-1} - (\delta + n + g) = 0$

- Golden rule level of capital:
  - ${k^*}_{GR} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$

The importance of the savings rate for steady-state output

- Recall the equation for steady-state output in case of a Cobb-Douglas production function:
  - $y^* = \left(\frac{s}{\delta+n+g}\right)^{\alpha/(1-\alpha)}$
- The elasticity of steady-state output w.r.t. the savings rate equals $\alpha/(1-\alpha)$
  - $\partial y^*/\partial s = \alpha/(1-\alpha)\left(\frac{s}{\delta+n+g}\right)^{\alpha/(1-\alpha)} s^{-1} \quad \rightarrow$
  - $(\partial y^*/\partial s)(s/y^*) = \alpha/(1-\alpha)$

The importance of the savings rate for steady-state output

- If α =1/3, the elasticity of steady-state output w.r.t. the savings rate is 1/2
- The elasticity of steady-state output w.r.t. the savings rate is lower, the lower is the income share of capital
- This is for two reasons:
  - The savings curve is more concave, the smaller is α
  - The effect of a change in the steady-state stock of capital (per effective labour) upon steady-state output is lower, the smaller is α
- Romer, section 1.5 establishes the same result for the case of a more general production function (not necessarily a Cobb-Douglas one)

The length of the transition process

- Based on a first-order Taylor approximation, it can be derived that the gap between k and its steady-state value decreases at a constant rate λ,
  - Where $\lambda = (1 - \alpha)(\delta + n + g)$
  - The speed of adjustment of y equals that of k
- Take α=1/3; δ=3%; n=1% and g=2%:
  - λ=4% per year

The length of the transition process

- This implies that it takes about 17 years for the gap between k and its steady-state value to have halved
  - (Calculate *t* from $\exp(-\lambda t) = 0.5$)
- After 35 years, the gap is still a quarter of its original value
  - (Calculate *t* from $\exp(-\lambda t) = 0.25$)